ENGR 105: Feedback Control Design
Winter 2013

Lecture 7 - Sinusoidal Inputs

Wednesday, January 23, 2013

Today's Objectives

1. derive the Laplace transform of a sinusoid
2. find the time response of a general transfer function to a sinusoidal input
3. show an example for an RC circuit

Reading: FPE Section 3.1

Sinusoidal inputs

The transfer function of a system is the Laplace transform of the system's impulse response. We have shown that this makes it very easy to obtain the impulse response of a system. Using partial fraction expansion, it is also straightforward to solve for the response to any general input. Sinusoidal inputs offer another interesting interpretation of the transfer function.

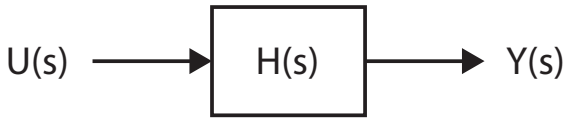
1 Laplace transform of a sinusoid

We know that poles at $s = \pm j\omega$ are associated with sinusoids. In particular,

Similarly,

2 Time response to sinusoidal inputs

Consider a stable system $H(s)$ and sinusoidal input of the form:



So after the response from the stable poles dies out,

The output of the system therefore becomes some sort of sinusoid. To be more specific, we can evaluate the residues as follows:

$H(j\omega)$ is just a complex number for any value of ω. It can be considered in terms of real and imaginary components, or a magnitude and phase:

Putting this together gives the time response:

So $H(j\omega)$ is a complex number representing the change in magnitude and phase experienced by a sinusoidal input of frequency ω.

Another way to get the same result is to use convolution:

$$\begin{aligned} y(t) &= \int_0^\infty h(\tau)u(t-\tau)\ d\tau \\ &= \int_0^\infty h(\tau)e^{\zeta(t-\tau)}\ d\tau \quad \text{if } u(t) = e^{\zeta t} \\ &= e^{\zeta t}\int_0^\infty h(\tau)e^{-\zeta t}\ d\tau \\ &= e^{\zeta t}H(\zeta) \end{aligned}$$

If $u(t)$ is a sinusoid, it can be written as

$$u(t) = \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t}$$

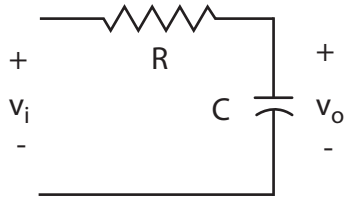
which gives

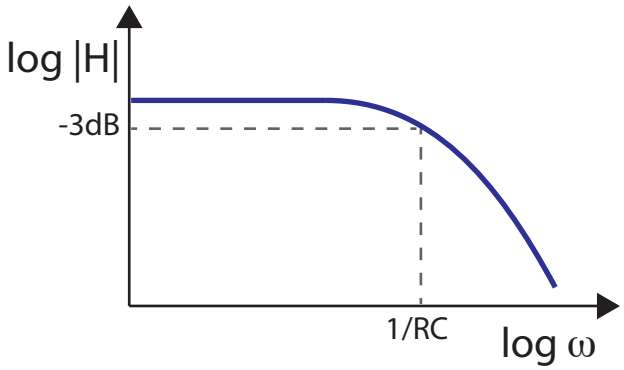
$$y(t) = \frac{A}{2j}H(j\omega)e^{j\omega t} - \frac{A}{2j}H(-j\omega)e^{-j\omega T}$$

which can be simplified as above.

3 RC Circuit Example

What is the frequency response of this RC circuit?





A sinusoid at frequency $\omega_c = \frac{1}{RC}$ has been shifted in phase by $45°$ and reduced in magnitude to $\frac{\sqrt{2}}{2}$ relative to the input.