Problem 1)

1) Equations

$$0 = ml\ddot{\theta}_1 + mg\theta_1 + \frac{9}{16}kl(\theta_1 - \theta_2)$$

$$0 = ml\ddot{\theta}_2 + mg\theta_2 + \frac{9}{16}kl(\theta_2 - \theta_1)$$

$$ml\left[s^2\theta_1(s) - s\,\theta_1(0) - \theta_1'(0)\right] + mg\,\theta_1(s) + \frac{9}{16}kl\left(\theta_1(s) - \theta_2(s)\right) = 0$$

$$\Rightarrow ① \left(mls^2 + mg + \frac{9}{16}kl\right)\theta_1(s) - \frac{9}{16}kl\,\theta_2(s) = mls\,\theta_1(0) + ml\,\theta_1'(0)$$

$$ml\left[s^2\theta_2(s) - s\,\theta_2(0) - \theta_2'(0)\right] + \frac{9}{16}kl\left[\theta_2(s) - \theta_1(s)\right] = 0$$

$$\Rightarrow ② \left[-\frac{9kl}{16}\theta_1(s)\right] + \left(mls^2 + \frac{9}{16}kl\right)\theta_2(s) = mls\,\theta_2(0) + ml\,\theta_2'(0)$$

$$① \quad \alpha_1\theta_1(s) - \alpha_2\theta_2(s) = \alpha_3$$

$$② \quad \beta_1\theta_1(s) + \beta_2\theta_2(s) = \beta_3$$

$$① \Rightarrow \theta_1(s) = \frac{\alpha_3 + \alpha_2\theta_2(s)}{\alpha_1} \cdots ③ \rightarrow \text{Substitute in } ②$$

$$\left(\beta_2 + \frac{\beta_1\alpha_2}{\alpha_1}\right)\theta_2(s) = \left(\beta_3 - \frac{\alpha_3\beta_1}{\alpha_1}\right)$$

$$\Rightarrow \theta_2(s) = \frac{\beta_3\alpha_1 - \alpha_3\beta_1}{\beta_2\alpha_1 + \alpha_2\beta_1}$$

Substitute back in ③

$$\Rightarrow \theta_1(s) = \frac{\alpha_3\beta_2\alpha_1 + \cancel{\alpha_3\alpha_2\beta_1} + \alpha_2\beta_3\alpha_1 - \cancel{\alpha_3\alpha_2\beta_1}}{\alpha_1(\beta_2\alpha_1 + \alpha_2\beta_1)}$$

$$= \frac{\alpha_3\beta_2 + \alpha_2\beta_3}{\beta_2\alpha_1 + \alpha_2\beta_1}$$

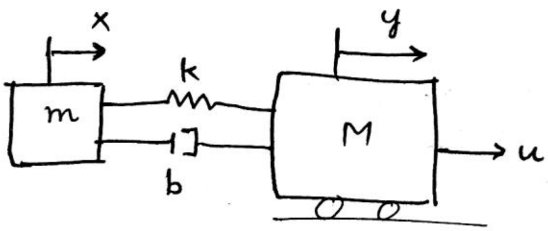
or

use cramer's rule

b) Cool feature:

Laplace transform transforms differential equations into algebraic equations. This is useful because it allows you to easily solve for $\theta_1(t)$ and $\theta_2(t)$ independently, even though the original differential equations were coupled. You would just need to take the inverse Laplace transform.

Problem 2)



$$M\ddot{y} = u - k(y-x) - b(\dot{y}-\dot{x})$$
$$m\ddot{x} = k(y-x) + b(\dot{y}-\dot{x})$$

Take Laplace Transform

$\cancel{(Ms^2+k)Y(s)}$

① $(Ms^2+bs+k)\,Y(s) - (bs+k)\,X(s) = U(s)$

② $(ms^2+bs+k)\,X(s) = (bs+k)\,Y(s)$

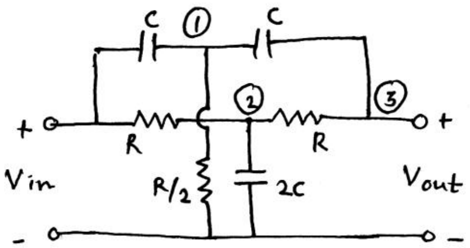
substitute $X(s)$ from ② in ①

$$\Rightarrow (Ms^2+bs+k)\,Y(s) - \frac{(bs+k)^2}{(ms^2+bs+k)}\,Y(s) = U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{(ms^2+bs+k)}{(Ms^2+bs+k)(ms^2+bs+k) - (bs+k)^2}$$

$$= \frac{(ms^2+bs+k)}{Mms^4 + (M+m)(bs+k)s^2}$$

Problem 3)



Write down current law equations for nodes 1, 2 & 3

1 $C\frac{d}{dt}(V_{in}-V_1) + \frac{0-V_1}{R/2} + C\frac{d}{dt}(V_{out}-V_1) = 0$

2 $\frac{V_{in}-V_2}{R} + 2C\frac{d}{dt}(0-V_2) + \frac{V_{out}-V_2}{R} = 0$

3 $C\frac{d}{dt}(V_1 - V_{out}) + \frac{V_2 - V_{out}}{R} = 0$

We need to eliminate V_1, V_2 from three equations and find the relation between V_{in} & V_{out}

$$V_1 = \frac{Cs}{2\left(Cs+\frac{1}{R}\right)}(V_{in}+V_{out})$$

$$V_2 = \frac{\left(\frac{1}{R}\right)}{2\left(Cs+\frac{1}{R}\right)}(V_{in}+V_{out})$$

Substitute in any of the eqns.

$$\boxed{\frac{V_{out}}{V_{in}} = \frac{\left(s^2+\frac{1}{R^2C^2}\right)}{\left(s^2+\frac{4}{RC}s+\frac{1}{R^2C^2}\right)}}$$

Problem 4)

a) $f(t) = 1 + 2t$

from the tables,

$$F(s) = \frac{1}{s} + \frac{2}{s^2}$$

$$= \frac{(s+2)}{s^2}$$

b) $f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$

$$\Rightarrow F(s) = \left(\frac{1}{s+1}\right) + \frac{2}{(s+2)} + \frac{1}{(s+3)^2}$$

$$= \frac{(s+2)(s+3)^2 + 2(s+1)(s+3)^2 + (s+1)(s+2)}{(s+1)(s+2)(s+3)^2}$$

$$= \frac{(3s+4)(s^2+6s+9) + (s^2+3s+2)}{(s+1)(s+2)(s+3)^2}$$

$$= 3s^3$$

c) $F(s) = \dfrac{2(s^2+s+1)}{s(s+1)^2}$

carry out partial fraction expansion

$$F(s) = \frac{2(s^2+s+1)}{s(s+1)^2} = \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+1)^2}$$

$$C_1 = s\,F(s)\Big|_{s=0} = \frac{2(s^2+s+1)}{(s+1)^2}\Big|_{s=0} = 2$$

$$C_3 = (s+1)^2 F(s)\Big|_{s=(-1)} = \frac{2(s^2+s+1)}{s}\Big|_{s=-1} = (-2)$$

$$C_2 = \frac{d}{ds}\Big[(s+1)^2 F(s)\Big]_{s=-1}$$

$$= \frac{d}{ds}\Big[\frac{2(s^2+s+1)}{s}\Big]_{s=-1} = 0$$

c) contd..

$$\Rightarrow F(s) = \frac{2}{s} + \frac{0}{s+1} - \frac{2}{(s+1)^2}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2\{1 - te^{-t}\}1(t)$$

d) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0 \quad y(0) = 1 \quad \dot{y}(0) = 2$

Take laplace transform

$$s^2 Y(s) - s y(0) - \dot{y}(0) + sY(s) - y(0) + 3Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{s+3}{s^2+s+3}$$

$$= \frac{(s+\frac{1}{2}) + \frac{5}{2}}{(s+\frac{1}{2})^2 + \frac{11}{4}}$$

$$= \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \frac{11}{4}} + \frac{5\sqrt{11}}{11} \frac{\sqrt{\frac{11}{4}}}{(s+\frac{1}{2})^2 + \frac{11}{4}}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-\frac{1}{2}t} \cos \frac{\sqrt{11}}{2} t + \frac{5\sqrt{11}}{11} e^{-\frac{1}{2}t} \sin \frac{\sqrt{11}}{2} t$$

e) $\ddot{y}(t) + y(t) = t \quad y(0) = 1 \quad \dot{y}(0) = (-1)$

$$s^2 Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{s^3 - s^2 + 1}{s^2(s^2+1)}$$

$$= \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3 s + C_4}{s^2+1}$$

$$C_1 = \frac{d}{ds} \frac{(s^3 - s^2 + 1)}{(s^2+1)} \Big|_{s=0} = 0$$

$$C_2 = \left(\frac{s^3 - s^2 + 1}{s^2+1}\right)\Big|_{s=0} = 1$$

e) contd..

$$\Rightarrow \quad \frac{1}{s^2} + \frac{C_3 s + C_4}{s^2+1} = \frac{s^3 - s^2 + 1}{s^2(s^2+1)}$$

$$\Rightarrow \quad \frac{(s^2+1) + (C_3 s + C_4) s^2}{s^2(s^2+1)} = \frac{s^3 - s^2 + 1}{s^2(s^2+1)}$$

match co-efficients of like powers of s

$\Rightarrow \quad C_3 = 1$

$C_4 + 1 = -1 \quad \Rightarrow C_4 = (-2)$

$$\therefore Y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$

$$y(t) = t + \cos t - 2\sin t$$

Problem 5)

a) ① Impulse response $h_1(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

② Unit step response $h_2(t) = \mathcal{L}^{-1}\left(\frac{1}{s} H(s)\right) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$

③ Unit Ramp response $h_3(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2} H(s)\right) = \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!} = \frac{t^3}{6}$

none of them are <u>stable</u>. unbounded increase in response

b) ① Impulse response

$$h_1(t) = \mathcal{L}^{-1}\left(\frac{s+1}{s^2}\right) = (1+t)\,1(t)$$

② Unit step response

$$h_2(t) = \mathcal{L}^{-1}\left(\frac{1}{s} \cdot \frac{s+1}{s^2}\right) = t + \frac{t^2}{2}$$

③ Unit Ramp response

$$h_3(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2} \cdot \frac{s+1}{s^2}\right) = \frac{t^2}{2} + \frac{t^3}{6}$$

none of them are <u>stable</u>. response $\rightarrow \infty$ as $t \rightarrow \infty$

c) ① Step ~~Impulse~~ response

$$h_1(t) = \mathcal{L}^{-1}\left\{ \frac{(s-1)}{s(s+2)(s+1)} \right\}$$

$$\frac{(s-1)}{s(s+2)(s+1)} = \frac{c_1}{s} + \frac{c_2}{(s+2)} + \frac{c_3}{(s+1)}$$

$$c_1 = \left.\frac{(s-1)}{(s+2)(s+1)}\right|_{s=0} = \left(-\frac{1}{2}\right)$$

$$c_2 = \left.\frac{(s-1)}{s(s+1)}\right|_{s=(-2)} = \frac{(-3)}{(-2)(-1)} = \left(-\frac{3}{2}\right)$$

$$c_3 = \left.\frac{(s-1)}{s(s+2)}\right|_{s=(-1)} = \frac{(-2)}{(-1)(1)} = (2)$$

$$\Rightarrow h_2(t) = -\frac{1}{2}1(t) - \frac{3}{2}e^{-2t} + 2e^{-t}$$

② Impulse ~~Step~~ response

$$h_1(t) = \mathcal{L}^{-1}\left\{ \frac{(s-1)}{(s+2)(s+1)} \right\}$$

$$\frac{s-1}{(s+2)(s+1)} = \frac{c_1}{(s+2)} + \frac{c_2}{(s+1)}$$

$$c_1 = \left.\frac{s-1}{s+1}\right|_{s=-2} = \frac{(-3)}{(-1)} = 3$$

$$c_2 = \left.\frac{(s-1)}{(s+2)}\right|_{s=-1} = (-2)$$

$$h_1(t) = 3e^{-2t} - 2e^{-t}$$

③ ~~Step~~ response
Ramp

$$h_3(t) = \mathcal{L}^{-1}\left\{ \frac{(s-1)}{s^2(s+2)(s+1)} \right\}$$

$$\frac{(s-1)}{s^2(s+2)(s+1)} = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{(s+2)} + \frac{c_4}{(s+1)}$$

$$c_2 = \left.\frac{(s-1)}{(s+2)(s+1)}\right|_{s=0} = \left(-\frac{1}{2}\right) \qquad c_3 = \frac{(-3)}{(4)(-1)} = \frac{3}{4} \qquad c_4 = \left(\frac{-2}{1}\right) = (-2)$$

~~compare constant term of numerator~~

$$c_1 = \frac{d}{ds}\left(s^2 F(s)\right)_{s=0}.$$

$$= \frac{d}{ds}\left(\frac{s-1}{s^2+3s+2}\right)\Bigg|_{s=0}$$

$$= \frac{(s^2+3s+2)-(s-1)(2s+3)}{(s^2+3s+2)^2}\Bigg|_{s=0}$$

$$\Rightarrow c_1 = \frac{2+3}{4} = \frac{5}{4}$$

$$\therefore h_3(t) = \frac{5}{4}1(t) - \frac{1}{2}t + \frac{3}{4}e^{-2t} - 2e^{-t}$$

d) None of the responses in parts a & b are stable only the step and impulse response of part c are stable

Problem 8)

a) $(I + m_p l^2)\ddot{\theta}' - m_p g l \theta' = m_p l \ddot{x}$ ①

$(m_t + m_p)\ddot{x} + b\dot{x} - m_p l \ddot{\theta}' = u$ ②

② $\Rightarrow \left((m_t + m_p)s^2 + bs\right) X(s) - m_p l s^2 \theta'(s) = U(s)$

① $\Rightarrow (-m_p l s^2) X(s) + \left[(I + m_p l^2)s^2 - m_p g l\right] \theta'(s) = 0$

$\therefore \quad X(s) = \dfrac{\left[(I + m_p l^2)s^2 - m_p g l\right]}{m_p l s^2} \theta'(s)$

Substitute in ②

$$\Rightarrow \frac{\left[(m_t + m_p)\cancel{s^2} + b\cancel{s}\right]\left[(I + m_p l^2)s^2 - m_p g l\right]}{m_p l \cancel{s^2}} \theta'(s) - m_p l s^2 \theta'(s) = U(s)$$

$$\Rightarrow \frac{\theta'(s)}{U(s)} = \frac{(m_p l s)}{\left[(m_t + m_p)s + \underbrace{\cancel{b}}_{0}\right]\left[(I + m_p l^2)s^2 - m_p g l\right] - m_p^2 l^2 s^3}$$

$$= \frac{m_p l}{\left[(I + m_p l^2)s^2 - m_p g l\right](m_t + m_p) - m_p^2 l^2 s^2}$$

$$= \frac{m_p l}{\left[(I + m_p l^2)(m_t + m_p) - m_p^2 l^2\right]s^2 - m_p g l (m_t + m_p)}$$

b) $(\theta_r(s) - \theta'(s))\, D(s)\, G(s) = \theta'(s)$

$\Rightarrow \theta_r(s)\, D(s)\, G(s) = \theta'(s)\left[1 + D(s)\, G(s)\right]$

$$\Rightarrow \frac{\theta'(s)}{\theta_r(s)} = \frac{D(s)\, G(s)}{1 + D(s)\, G(s)}$$

c) $$\frac{\theta'(s)}{\theta_r(s)} = \frac{K_p m_p l}{[(I+m_p l^2)(m_t+m_p)-m_p^2 l^2]s^2 - m_p g l(m_t+m_p) + k_p m_p}$$

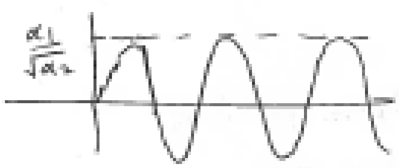
Reference

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2} \qquad \mathcal{L}(e^{-at}) = \frac{1}{s+a}$$

$$\frac{\theta'(s)}{\theta_r(s)} = \frac{\alpha_1}{s^2+\alpha_2} \quad \text{or} \quad \frac{\alpha_1}{s^2-\alpha_2}$$

$\boxed{\alpha_2 > 0}$

$\mathcal{L}^{-1}\left(\frac{\alpha_1}{s^2+\alpha_2}\right)$ is of the form $\frac{\alpha_1}{\sqrt{\alpha_2}}\sin(\sqrt{\alpha_2}\,t)$



oscillating marginally ~~(unstable)~~

$$\mathcal{L}^{-1}\left(\frac{\alpha_1}{s^2-\alpha_2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{\alpha_1}{2\sqrt{\alpha_2}}\left\{\frac{1}{s-\sqrt{\alpha_2}} - \frac{1}{s+\sqrt{\alpha_2}}\right\}\right)$$

$$= \frac{\alpha_1}{2\sqrt{\alpha_2}}\left\{\underbrace{e^{\sqrt{\alpha_2}\,t}}_{\text{This term will blow up as } t\to\infty \text{ (unstable)}} - e^{-\sqrt{\alpha_2}\,t}\right\}$$

$\Rightarrow$ cannot be stable with just k_p

(Note if $k_p = \frac{m_p g l(m_t+m_p)}{m_p l}$ then $\frac{\theta'(s)}{\theta_r} = \frac{\alpha_3}{s^2} \rightarrow$ still unstable

d) Replace k_p with $(k_p + k_d s)$

$$\frac{\theta'(s)}{\theta_r(s)} = \frac{(k_p+k_d s)\,m_p l}{[(I+m_p l^2)(m_t+m_p)-m_p^2 l^2]s^2 - m_p g l(m_t+m_p) + k_p m_p l}$$

e) $$\frac{\theta'(s)}{\theta_r(s)} = \frac{(k_p + k_d s)}{3s^2 - 10(2) + (k_p + k_d s)}$$

$$= \frac{(k_p + k_d s)}{3s^2 + k_d s + (k_p - 20)}$$

for both closed loop poles to be at (-1)
denominator should be $3(s+1)^2$ (adjusting for leading co-efficient)

$\Rightarrow 3(s+1)^2 = 3s^2 + k_d s + (k_p - 20)$

compare coefficients of s

$\Rightarrow \boxed{k_d = 6 \text{ and } k_p = 23}$

f) step response $\Rightarrow \theta_r(s) = \frac{1}{s}$

using the values of k_p and K_d from part e

$$\frac{\theta'(s)}{\theta_r(s)} = \frac{(6s+23)}{s[3s^2 - 20 + 23 + 6s]} = \frac{(6s+23)}{3s(s+1)^2}$$

$\cancel{\theta'(s) =}$ We can use this to directly plot the step response using the control system toolbox in MATLAB or we can find the corresponding time response and plot it against a time vector in MATLAB

(I) using time response

$$\theta'(s) = \frac{(6s+23)}{3s(s+1)^2}$$

find inverse laplace using partial fractions

$$\theta'(t) = \frac{23}{3} - \frac{1}{3}e^{-t}(17t + 23)$$

note $\lim_{t \to \infty} \theta'(t) = \frac{23}{3}$ response would be stable and converge on to (23)

MATLAB code

```
t = [0:0.5:12];          % define a vector t
plot(t, 23/3-(3*exp(-t)).*(17*t+23))
```

(II) Directly use control system toolbox

```
G = tf([2  23/3], [1  2  1]);
step(G)
```

see attached plot

As expected the plot is stable and converges to $\frac{23}{3}$.
There are no oscillations as exponential decays faster than the linear term.