ENGR 105: Feedback Control Design
Winter 2013

Lecture 4 - The Transfer Function

Monday, January 14, 2013

Today's Objectives

1. introduce the concept of the transfer function
2. give the Laplace transform of the impulse response
3. use the impulse response as a basis for understanding the role of the transfer function
4. interpret the impulse response using the poles of the transfer function

Reading: FPE Section 3.1

1 The transfer function

The equations resulting from system modeling in this class take the form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^m u}{dt^m} + b_2 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_{m+1} u$$

Laplace transforming gives:

$$[s^n + a_1 s^{n-1} + \cdots + a_n]Y(s) = [b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}]U(s)$$

When initial conditions are set to zero, this can be arranged as a transfer function, or ratio of two polynomials, $H(s)$:

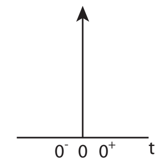
$$\frac{Y(s)}{U(s)} = H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

The transfer function can tell us many useful things about the system and has several interpretations.

2 The impulse response

The first interpretation of the transfer function that we will examine relates to the *impulse response*. An impulse is a signal that is nonzero only at one point in time.

Unit impulse:



$\delta(t) = 0$ except when $t = 0$

$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0-}^{0+} \delta(t)dt \equiv 1$

What is the Laplace transform of an impulse?

$\mathcal{L}[\delta(t)] = \int_{0-}^{\infty} \delta(t)e^{-st}dt = \int_{0-}^{0+} \delta(t)e^{0}dt = \int_{0-}^{0+} \delta(t)dt = 1$

Remember the step function:

$1(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$ $\qquad \frac{d}{dt}1(t) = \delta(t)$

$\mathcal{L}[1(t)] = \frac{1}{s}$ $\qquad \mathcal{L}[\frac{d}{dt}1(t)] = s\,\mathcal{L}[1(t)] - 1(0) = s \cdot \frac{1}{s} - 0 = 1$

The impulse is the derivative of the step. Everything we know about differentiation checks out.

3 Understanding the Laplace transform in terms of the impulse response

Back to the transfer function:

If $\frac{Y(s)}{U(s)} = H(s)$, and $U(s)$ is an impulse, then the impulse response is $Y(s) = H(s)$.

Thus, $\mathcal{L}^{-1}[H(s)] = h(t) = y(t)$ for an impulse.

$\Rightarrow$ So in general, the transfer function is the Laplace transform of the system response to an impulse.

Think of the impulse response as figuratively (and sometimes literally) hitting the system with a hammer. If you look at the time response, you can see a lot of things about a system, including its stability, frequency of any resonances, decay time, etc. We can similarly see all of these in the transfer function if we know where to look.

Why do we the transfer function at all? That is, why focus on $G(s)$ and not $g(t)$ directly? It is usually *much* easier to solve problems in the *Laplace domain* (also sometimes called the *frequency domain* or *s-domain*), since multiplication in s is convolution in t.

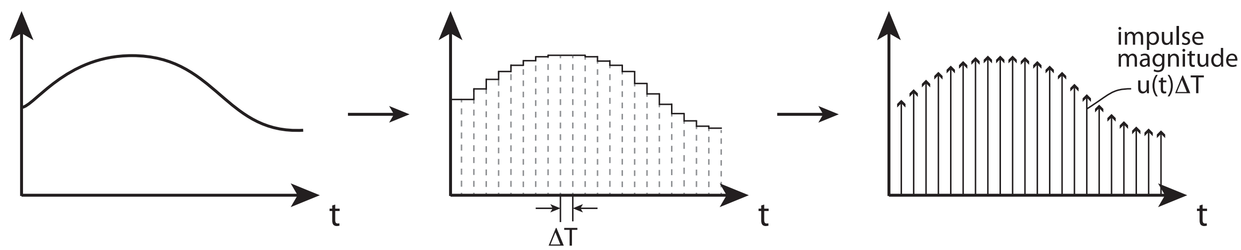
Laplace

$y(t) = \mathcal{L}^{-1}[H(s)U(s)]$

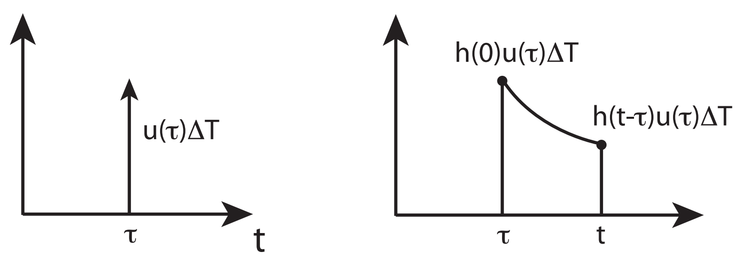
Time domain

$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$

What this says is that we can think of the system response as a composition of impulse responses.



Think of $u(t)$ as a train of impulses. What is the response to these impulses?



At any time t, the output y us a result of all past impulses, so:

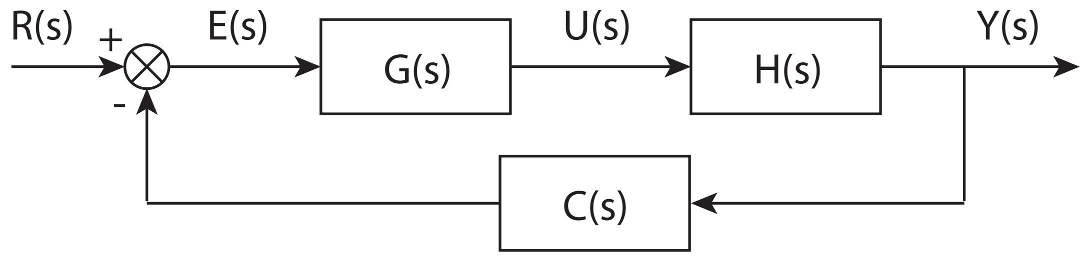
$y(t) = [h(t)u(0) + h(t-\Delta T)u(\Delta T) + h(t-2\Delta T)u(2\Delta T) + \ldots]\Delta T$

where $h(t)$ is the response to an impulse after t seconds.

In the limit: $y(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau = u(t) * h(t)$

where $*$ is the symbol for convolution.

Clearly, we can do this convolution in time domain, but it is harder than multiplication and may need to be repeated a number of times, for example, in a feedback control system:



$Y(s) = G(s)H(s)E(s)$

$E(s) = R(s) - C(s)Y(s)$

$E(s) = R(s) - C(s)G(s)H(s)E(s)$
$\nwarrow$ This would be quite a lot of convolution!

$\Rightarrow \frac{E(s)}{R(s)} = \frac{1}{1+C(s)G(s)H(s)}$

The transfer function from $R(s)$ to $E(s)$ is much easier to compute with multiplication than convolution. And the key to the block diagram being mathematically correct, where signals are multiplied by the blocks through which they pass, is that the Laplace transformation takes convolution in the time domain and converts it to multiplication in the s-domain.

Since the transfer function is the Laplace transform of the impulse response, it should have the same information contained in it. How do we extract that information?

4 Poles and the impulse response

One way to understand the impulse response is to do a partial fraction expansion of the transfer function.

The transfer function can be written as:

$$H(s) = \frac{K \prod^{m}(s-z_i)}{\prod^{n}(s-p_i)}$$

z_i are the zeros and p_i are the poles. $m \leqslant n$ for a physical system.

It can also be written as

$H(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}$

(This form requires that all the poles be distinct – we will look at repeated poles shortly.)

The impulse response is therefore given by:

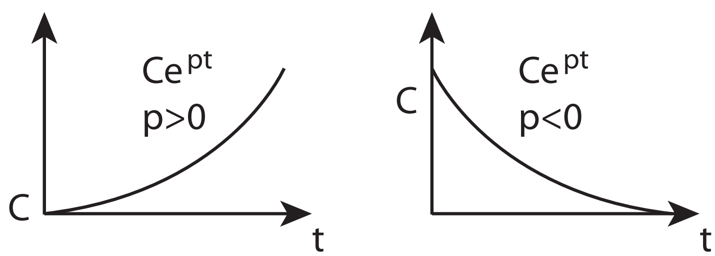
$y(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}\right]$

$y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \cdots + C_n e^{p_n t}$

Just a sum of our exponential building blocks! We can tell a lot about the response just by knowing the poles of the system.

Real and complex poles

If a pole is real, it must be negative for the system to be stable. The exponential can only grow or decay.



What about a pole $p = -\sigma + j\omega$?

$Ce^{(-\sigma+j\omega)t} = Ce^{-\sigma t}e^{j\omega t} = Ce^{-\sigma t}[\cos\omega t + j\sin\omega t]$

$\nearrow$ decays or grows $\quad\uparrow$ oscillates $\quad\nwarrow$ how do we handle j?

If there is a pole $p_1 = -\sigma + j\omega$, its complex conjugate $p_2 = -\sigma - j\omega$ must also be a pole. Then:

$$(s-p_1)(s-p_2) = s^2 - (-\sigma + j\omega)s + (-\sigma - j\omega)s + \sigma^2 + \omega^2$$

$$= s^2 + 2\sigma s + \sigma^2 + \omega^2$$

The j terms cancel out, and we are left with only real coefficients. (whew!)

This is also true of the coefficients in the partial fraction expansion: if $C_1 = \alpha - j\beta$ then $C_1 = \alpha + j\beta$ must also be a coefficient so that the numerator polynomial has real coefficients.

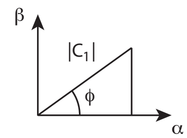
This means that a complex pair of poles appears in the impulse response as:

$$H(s) = \frac{\alpha - j\beta}{s+\sigma-j\omega} + \frac{\alpha+j\beta}{s+\sigma+j\omega} + \dots$$

So $y(t) = (\alpha - j\beta)e^{-\sigma t}[\cos\omega t + j\sin\omega t] + (\alpha + j\beta)e^{-\sigma t}[\cos\omega t - j\sin\omega t] + \dots$

$$= e^{-\sigma t}[2\alpha\cos\omega t + 2\beta\sin\omega t] + \dots$$

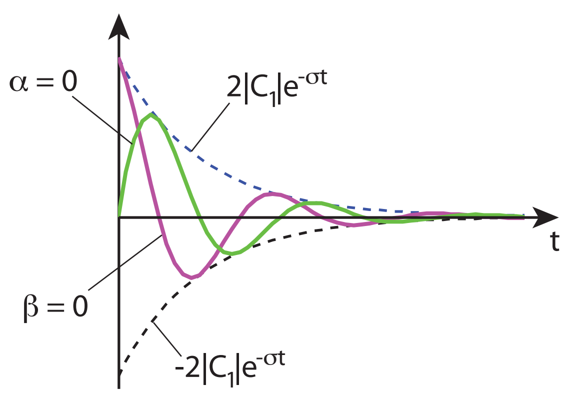
$$= 2|C_1|e^{-\sigma t}\cos(\omega t - \phi) + \dots$$



where $|C_1| = \sqrt{\alpha^2 + \beta^2}$ and $\tan\phi = \frac{\beta}{\alpha}$

Why?
$\alpha = |C_1|\cos\phi$ and $\beta = |C_1|\sin\phi$
so $\alpha\cos\omega t + \beta\sin\omega t = |C_1|\cos\phi\cos\omega t + |C_1|\sin\phi\sin\omega t = |C_1|\cos(\omega t - \phi)$

Each value has some meaning:
σ represents the rate of exponential decay
ω represents the frequency of oscillation
α represents how much the cosine term exists
β represents how much the sine term exists




```
t = 0:0.1:5;
sigma = 1;
omega = pi;

% case where beta = 0
alpha = 1;
beta = 0;
C1 = sqrt(alpha^2 + beta^2);
phi = atan2(beta, alpha);
y1 = 2*C1*exp(-sigma.*t).*cos(omega.*t - phi);
env_pos = 2*C1*exp(-sigma.*t);
env_neg = -2*C1*exp(-sigma.*t);
plot(t, env_pos, '--b', t, env_neg, '--k', t, y1, '-m')

% case where alpha = 0
alpha = 0;
beta = 1;
C1 = sqrt(alpha^2 + beta^2);
phi = atan2(beta, alpha);
y2 = 2*C1*exp(-sigma.*t).*cos(omega.*t - phi);
hold on
plot(t, y2, '-g')

ylabel('y')
xlabel('time')
legend('positive envelope', 'negative envelope', ...
'y(t) for \beta = 0', 'y(t) for \alpha = 0')
```

Real poles give a stable response when they are negative.
Complex poles give a stable response when they have negative real parts.

If $\sigma = 0$, $s = \pm j\omega$, then $y(t) = 2|C_1|\cos(\omega t - \phi) \Rightarrow$ a non-decaying sinusoid.