ENGR 105: Feedback Control Design
Winter 2013

Lecture 7 - Sinusoidal Inputs

Wednesday, January 23, 2013

Today's Objectives

1. derive the Laplace transform of a sinusoid
2. find the time response of a general transfer function to a sinusoidal input
3. show an example for an RC circuit

Reading: FPE Section 3.1

Sinusoidal inputs

The transfer function of a system is the Laplace transform of the system's impulse response. We have shown that this makes it very easy to obtain the impulse response of a system. Using partial fraction expansion, it is also straightforward to solve for the response to any general input. Sinusoidal inputs offer another interesting interpretation of the transfer function.

1 Laplace transform of a sinusoid

We know that poles at $s = \pm j\omega$ are associated with sinusoids. In particular,

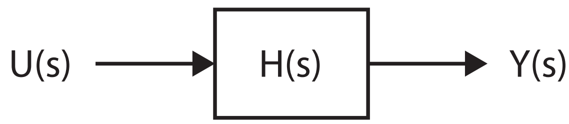
$$\begin{aligned}
\mathcal{L}[\sin\omega t] &= \int_0^\infty (\sin\omega t)e^{-st}\ \mathrm{d}t \\
&= \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st}\ \mathrm{d}t \\
&= \frac{1}{2j}\int_0^\infty \left[e^{(j\omega - s)t} - e^{-(j\omega + s)t}\right]\ \mathrm{d}t \\
&= \frac{1}{2j}\left[\frac{-1}{j\omega - s} + \frac{1}{-j\omega - s}\right] \\
&= \frac{1}{2j}\left[\frac{2j\omega}{(j\omega - s)(-j\omega - s)}\right] \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

Similarly,

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2 + \omega^2}$$

2 Time response to sinusoidal inputs

Consider a stable system $H(s)$ and sinusoidal input of the form: $U(s) = \frac{\omega A}{s^2+\omega^2}$



$$Y(s) = \underbrace{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + ... + \frac{a_n}{s-p_n}}_{\text{poles of } H(s) \text{ are stable so response dies out as } t \to \infty} + \underbrace{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}}_{\text{poles from the input oscillate without decaying}}$$

So after the response from the stable poles dies out,

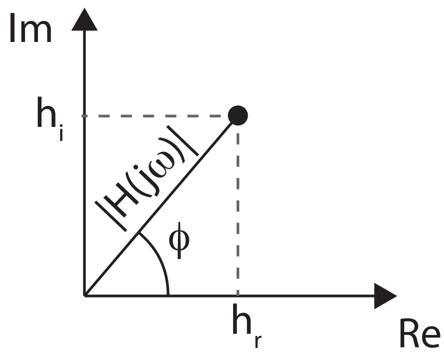
$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

The output of the system therefore becomes some sort of sinusoid. To be more specific, we can evaluate the residues as follows:

$$\begin{aligned}
a &= (s+j\omega)Y(s)|_{s=-j\omega} \\
&= (s+j\omega)H(s)\frac{\omega A}{s^2+\omega^2}\bigg|_{s=-j\omega} \\
&= H(s)\frac{\omega A}{s-j\omega}\bigg|_{s=-j\omega} \\
&= -\frac{1}{2j}H(-j\omega)A
\end{aligned}$$

$$\begin{aligned}
\bar{a} &= (s-j\omega)H(s)\frac{\omega A}{s^2+\omega^2}\bigg|_{s=-j\omega} \\
&= \frac{1}{2j}H(j\omega)A
\end{aligned}$$

$H(j\omega)$ is just a complex number for any value of ω. It can be considered in terms of real and imaginary components, or a magnitude and phase:



$$H(j\omega) = |H(j\omega)|(\cos\phi + j\sin\phi)$$

$$= |H(j\omega)|e^{j\phi}$$

$$H(-j\omega) = \bar{H}(j\omega) = |H(j\omega)|e^{-j\phi}$$

Putting this together gives the time response:

$$y(t) = -\frac{1}{2j}|H(j\omega)|Ae^{-j\phi}e^{-j\omega t} + \frac{1}{2j}|H(j\omega)|Ae^{j\phi}e^{j\omega t}$$

$$= A|H(j\omega)|\frac{1}{2j}\left[e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right]$$

$$= \underset{\substack{\uparrow \\ \text{input} \\ \text{amplitude}}}{A} \quad \underset{\substack{\uparrow \\ \text{change} \\ \text{in} \\ \text{amplitude}}}{|H(j\omega)|} \quad \sin(\underset{\substack{\uparrow \\ \text{input} \\ \text{frequency}}}{\omega t} \; + \; \underset{\substack{\uparrow \\ \text{phase} \\ \text{shift}}}{\phi})$$

So $H(j\omega)$ is a complex number representing the change in magnitude and phase experienced by a sinusoidal input of frequency ω.

$$u = A\sin\omega t \rightarrow \boxed{H(j\omega)} \rightarrow y = A|H(j\omega)|\sin(\omega t + \phi)$$

Another way to get the same result is to use convolution:

$$\begin{aligned} y(t) &= \int_0^\infty h(\tau)u(t-\tau)\ d\tau \\ &= \int_0^\infty h(\tau)e^{\zeta(t-\tau)}\ d\tau \quad \text{ if } u(t) = e^{\zeta t} \\ &= e^{\zeta t}\int_0^\infty h(\tau)e^{-\zeta t}\ d\tau \\ &= e^{\zeta t}H(\zeta) \end{aligned}$$

If $u(t)$ is a sinusoid, it can be written as

$$u(t) = \frac{A}{2j} e^{j\omega t} - \frac{A}{2j} e^{-j\omega t}$$

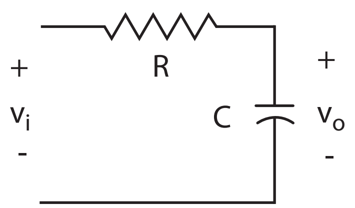
which gives

$$y(t) = \frac{A}{2j} H(j\omega) e^{j\omega t} - \frac{A}{2j} H(-j\omega) e^{-j\omega T}$$

which can be simplified as above.

3 RC Circuit Example

What is the frequency response of this RC circuit?



$$\frac{V_0(s)}{V_i(s)} = H(s) = \frac{1}{RCs+1}$$

$$H(j\omega) = \frac{1}{RCj\omega + 1} = \frac{1 - j\omega RC}{1 - \omega^2 R^2 C^2}$$

$$|H(j\omega)| = \sqrt{\left(\frac{1}{1 + \omega^2 R^2 C^2}\right)^2 + \left(\frac{\omega RC}{1 + \omega^2 R^2 C^2}\right)^2}$$

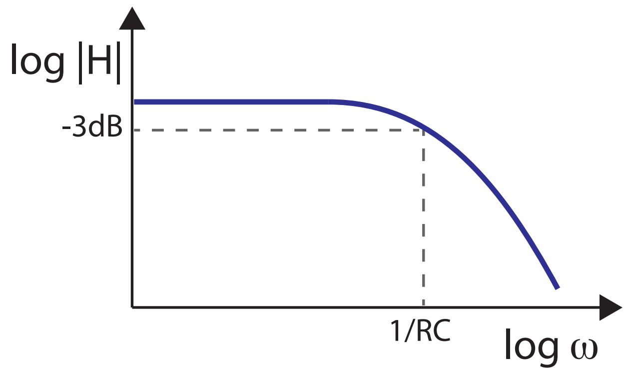
$|H(j\omega)| = 1$ at $\omega = 0 \;\Rightarrow$ DC gain of 1

$|H(j\omega)| \to 0$ as $\omega \to 0 \;\Rightarrow$ A low pass filter.

When $\omega_c = \frac{1}{RC}$:

$|H(j\omega_c)| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$ and $H(j\omega_c) = \frac{1}{2} - \frac{1}{2}j,$

Hence, $\phi = -45°$



A sinusoid at frequency $\omega_c = \frac{1}{RC}$ has been shifted in phase by 45° and reduced in magnitude to $\frac{\sqrt{2}}{2}$ relative to the input.