E105 Homework 3 Solutions

Problem 1

part a

The controlled input is input voltage, $v_s(t)$. The disturbances are thickness of the incoming material, $T(t)$. The output is the thickness of the outgoing material $x(t)$.

part b

A force balance on the top roller gives:

$$\begin{aligned} m\ddot{x}(t) &= c(T(t) - x(t)) - F_m(t) - b\dot{x} \\ ms^2X(s) &= c(T(s) - X(s)) - F_m(s) - bsX(s) \\ \left(ms^2 + bs + c\right)X(s) &= cT(s) - F_m(s) \\ X(s) &= \frac{c}{ms^2 + bs + c}T(s) - \frac{1}{ms^2 + bs + c}F_m(s) \end{aligned}$$

Here we ignore initial conditions because we are only constructing a block diagram, we do not include initial conditions in block diagrams. A torque balance on the final gear (the "rack and pinion") gives

$$\begin{aligned} RF_m(t) &= NT_{\text{motor}} \\ &= NK_t i_0(t) \end{aligned}$$

where T_{motor} is the torque due to the motor on the first gear, multiplied by N to account for the gear ratios between the first and last gears. Laplace:

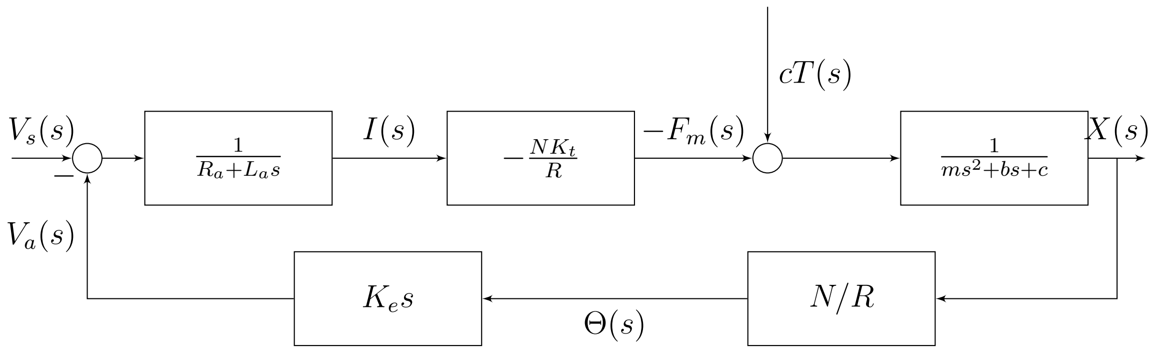
$$\begin{aligned} RF_m(s) &= NK_t I_0(s) \\ F_m(s) &= \frac{NK_t}{R} I_0(s) \end{aligned}$$

The kinematics of the roller/gear gives

$$\begin{aligned} x(t) &= \theta(t)R/N \\ X(s) &= R/N\Theta(s) \end{aligned}$$

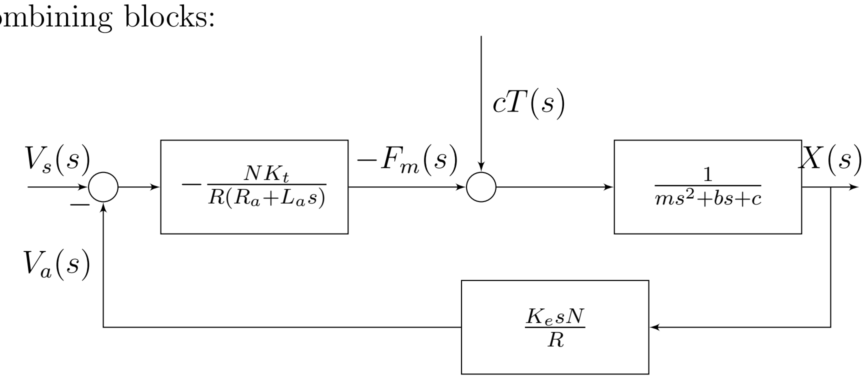
where $\theta(t)$ is the angle of the final gear. Finally, summing voltages around the circuit:

$$\begin{aligned} v_s(t) - i_0(t)R_a - L_a\frac{di_0(t)}{dt} - v_a(t) = 0 \\ V_s(s) - I_0(s)R_a - L_a sI(s) - V_a(s) = 0 \\ V_s(s) - V_a(s) = (R_a + L_a s)I_0(s) \\ I_0(s) = \frac{1}{R_a + L_a s}V_s(s) - \frac{1}{R_a + L_a s}V_a(s) \end{aligned}$$

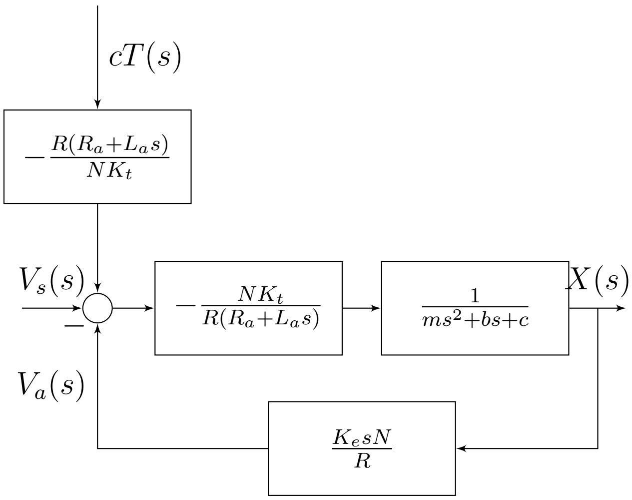


part c

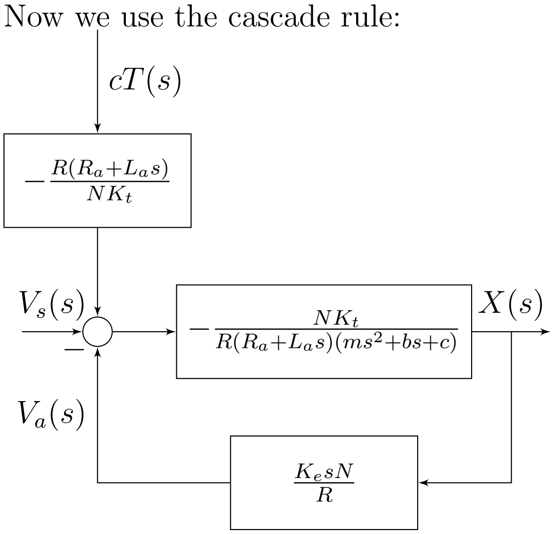
Combining blocks:



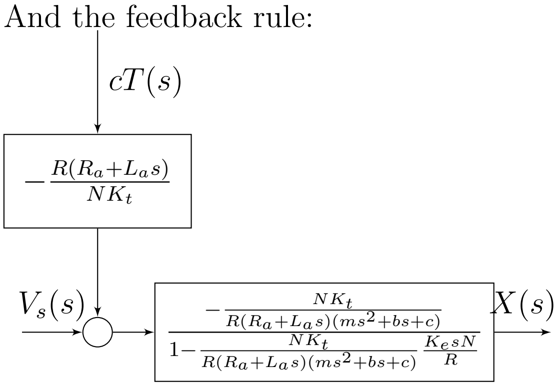
Now we can shift the disturbance to the left:



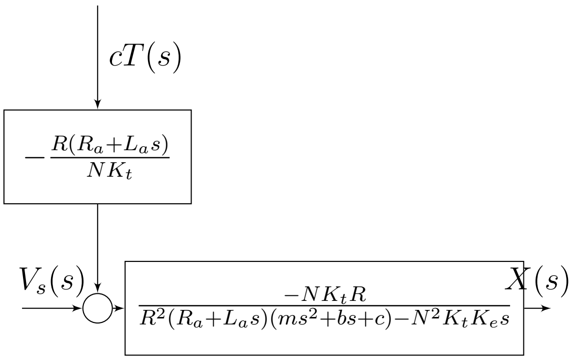
Now we use the cascade rule:



And the feedback rule:



And simplify the expressions in the blocks:



Problem 2

From inspection, we have

$$X_2(s) = \tfrac{1}{s}X_1(s) \qquad\qquad X_3(s) = \tfrac{1}{s}X_2(s)$$

which is equivalent to

$$sX_2(s) = X_1(s) \qquad\qquad s^2X_3(s) = X_1(s)$$

Then

$$\begin{aligned}
\tfrac{1}{s}\big(U(s) - a_1X(s) - a_2X_2(s) - a_3X_3(s)\big) &= X_1(s) \\
U(s) - a_1X(s) - a_2\tfrac{1}{s}X_1(s) - a_3\tfrac{1}{s^2}X_1(s) &= sX_1(s) \\
U(s) &= \big(a_1 + a_2\tfrac{1}{s} + a_3\tfrac{1}{s^2} + s\big)X_1(s) \\
\frac{1}{a_1 + a_2\frac{1}{s} + a_3\frac{1}{s^2} + s}U(s) &= X_1(s)
\end{aligned}$$

Then finally the output is

$$\begin{aligned}
Y(s) &= b_1X_1(s) + b_2X_2(s) + b_3X_3(s) \\
Y(s) &= b_1X_1(s) + b_2\tfrac{1}{s}X_1(s) + b_3\tfrac{1}{s^2}X_1(s) \\
Y(s) &= \big(b_1 + b_2\tfrac{1}{s} + b_3\tfrac{1}{s^2}\big)X_1(s) \\
Y(s) &= \frac{b_1 + b_2\frac{1}{s} + b_3\frac{1}{s^2}}{a_1 + a_2\frac{1}{s} + a_3\frac{1}{s^2} + s}U(s) \\
Y(s) &= \frac{b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}U(s)
\end{aligned}$$

Problem 3

part a

- The poles of the given system are 0, $-p$, and $-\zeta\omega_n \pm j\omega_n\sqrt{\zeta^2 - 1}$, so we must have $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{\zeta^2 - 1}$.

- To find α_1:

$$sY(s)|_{s=0} = \alpha_1 = \frac{\omega_n^2 p}{\omega_n^2 p} = 1$$

- To find α_2:

$$\begin{aligned}(s+p)Y(s)|_{s=-p} = \alpha_2 &= \left.\frac{\omega_n^2 p}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right|_{s=-p} \\ &= \frac{-\omega_n^2}{p^2 - 2\zeta\omega_n p + \omega_n^2}\end{aligned}$$

part b

First note that

$$\Re(\alpha) = |\alpha_3|\cos(\angle\alpha_3) \qquad\qquad \Im(\alpha) = |\alpha_3|\sin(\angle\alpha_3)$$

and that

$$\cos(u - v) = \cos(u)\cos(v) + \sin(u)\sin(v)$$

Combining these, taking $u = \angle\alpha$, we get

$$\cos(\angle\alpha - v) = \cos(\angle\alpha)\cos(v) + \sin(\angle\alpha)\sin(v)$$

Multiplying by $|\alpha|$ and substituting,

$$|\alpha|\cos(\angle\alpha - v) = \Re(\alpha)\cos(v) + \Im(\alpha)\sin(v) \tag{1}$$

Returning to the original problem, and taking the inverse Laplace transform, using the identity $\mathcal{L}^{-1}(\frac{1}{s+a}) = e^{-at}$,

$$\begin{aligned}Y(s) &= \alpha_1 1(t) + \alpha_2 e^{-pt} + \alpha_3 e^{-(\sigma+j\omega)t} + \bar{\alpha}_3 e^{-(\sigma-j\omega)t} \\ &= \alpha_1 1(t) + \alpha_2 e^{-pt} + e^{-\sigma t}\left(\alpha_3 e^{-j\omega t} + \bar{\alpha}_3 e^{j\omega t}\right) \\ &= \alpha_1 1(t) + \alpha_2 e^{-pt} + e^{-\sigma t}\Big(\big(\Re(\alpha_3) + j\Im(\alpha_3)\big)e^{-j\omega t} + \big(\Re(\alpha_3) - j\Im(\alpha_3)\big)e^{j\omega t}\Big) \\ &= \alpha_1 1(t) + \alpha_2 e^{-pt} + e^{-\sigma t}\Big(\Re(\alpha_3)\big(e^{-j\omega t} + e^{j\omega t}\big) + j\Im(\alpha_3)\big(e^{-j\omega t} + e^{j\omega t}\big)\Big) \\ &= \alpha_1 1(t) + \alpha_2 e^{-pt} + 2e^{-\sigma t}\Big(\Re(\alpha_3)\cos(\omega t) + \Im(\alpha_3)\sin(\omega t)\Big)\end{aligned}$$

Now we use (1) to get

$$\begin{aligned}Y(s) &= \alpha_1 1(t) + \alpha_2 e^{-pt} + 2|\alpha_3|e^{-\sigma t}\cos(\omega t - \angle\alpha_3) \\ &= \alpha_1 1(t) + \alpha_2 e^{-pt} + 2|\alpha_3|e^{-\sigma t}\sin\left(\omega t - \angle\alpha_3 + \frac{\pi}{2}\right)\end{aligned}$$

So it must be that $\alpha_1 = 1$ (which is what we had from earlier), $A = \alpha_2$, $B = |\alpha_3|/2$, and $\theta = \angle\alpha_3 - \frac{\pi}{2}$.

part c

Note that

$$\lim_{p\to\infty} \alpha_2 e^{-pt} = 0$$

but

$$\lim_{p\to\infty} 1(t) > 0 \qquad\qquad \lim_{p\to\infty} |\alpha_3| e^{-\sigma t} \sin(\omega t + \angle\alpha_3) > 0$$

so these two terms dominate the response.

part d

Note that

$$\lim_{p\to 0} \alpha_2 = -1 \qquad\qquad \lim_{p\to 0} |\alpha_3| = 0$$

part e

The $1(t)$ and Ae^{-pt} terms dominate, because $\lim_{p\to 0} |\alpha_3| = 0$, while $e^{-\sigma t}\sin(\omega_d t + \theta)$ stays bounded.

part f

The slower response corresponds to the smaller poles. The code used was:

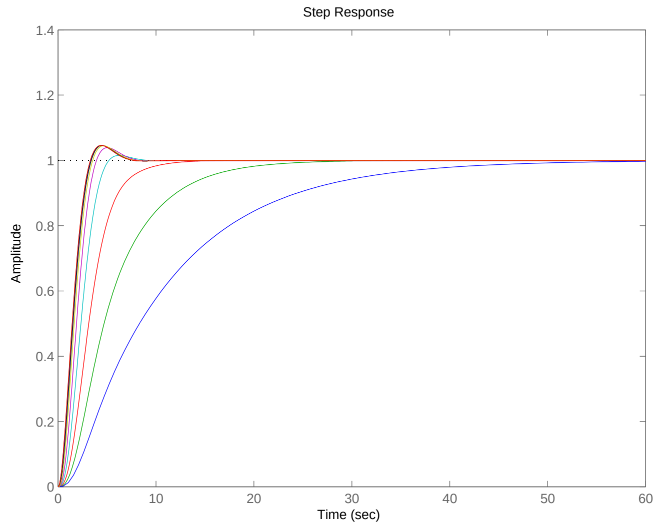
```
clf
hold all

omega=1;
zeta=.7;
P=logspace(-1,2,10);

% FIRST WAY
for p=P
    sys=tf(omega^2*p,conv([1,p],[1,2*zeta*omega,omega^2]));
    step(sys);
end

print -dpdf plot.pdf
%% SECOND WAY
clear; clf;

omega_d=omega*sqrt(1-zeta^2);
```




```
sigma=zeta*omega;
T=linspace(0,10,100);
for p=P
    alpha_2=-omega^2/(omega^2-2*p*zeta*omega+p^2);
    alpha_3_mag=p/(2*sqrt((1-zeta^2)*(p^2-2*p*zeta*omega+omega^2)));
    alpha_3_angle=atan2(omega_d,-sigma)+atan2(omega_d,p-sigma)+pi/2;
    Y=1+alpha_2*exp(-p.*T)+ ...
        2*alpha_3_mag*exp(-sigma.*T).*sin(omega_d.*T-alpha_3_angle);
    plot(T,Y)
end
```

Problem 4

part a

Applying the final value theorem,

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\
&= \lim_{s\to 0} sG(s)1/s \\
&= \lim_{s\to 0} \frac{s+2}{s+1} \\
&= 2
\end{aligned}$$

part b

The system is $G(s) = \frac{s+2}{s^2+s}$, so the response to a step is

$$\begin{aligned}
Y(s) &= \frac{s+2}{s^3+s^2} \\
&= \frac{s+2}{s^2(s+1)}
\end{aligned}$$

The original system had poles at 0 and -1, and are therefore not all in the open left half plane, so we suspect this might be unstable:

$$\begin{aligned}
Y(s) &= \frac{s+2}{s^2(s+1)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}
\end{aligned}$$

Then $B = s^2Y(s)|_{s=0} = 1$, so the output is $y(t) = A1(t) + t + Ce^{-t}$ and regardless of A and C, $\lim_{t\to\infty} = \infty$.

part c

The system is $G(s) = e^{-Ts}$, so the final value of the response to a step is

$$\begin{aligned}
\lim_{s\to 0} s\tfrac{1}{s}G(s) &= \lim_{s\to 0} s \cdot 1/s \cdot e^{-Ts} \\
&= 1
\end{aligned}$$

This is what we expect: The output is just a delayed step function, so the final value is 1.

part d

The system is $G(s) = \frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}}$, so the final value of the response to a step is

$$\begin{aligned}\lim_{s\to 0} s\tfrac{1}{s}G(s) &= \lim_{s\to 0} s \cdot 1/s \cdot \frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}} \\ &= 1\end{aligned}$$

Problem 5

For sinusoidal input $U(s) = \frac{\omega}{s^2+\omega^2}$ and output $Y(s) = G(s)U(s)$, we have

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} Y(s) \\ &= \lim_{s\to 0} sG(s)U(s) \\ &= \lim_{s\to 0} s \cdot \left(\frac{s^2+\omega^2}{s^2+2\zeta\omega s+\omega^2} \cdot \frac{\omega}{s^2+\omega^2}\right) \\ &= \lim_{s\to 0} \frac{\omega s}{s^2+2\zeta\omega s+\omega^2} \\ &= 0\end{aligned}$$

so the value of the output converges to 0.