ENGR 105: Feedback Control Design
Winter 2013

Lecture 9 - Performance Specifications

Monday, January 28, 2013

Today's Objectives

1. plot the step response of a second-order system
2. define several performance criteria

Reading: FPE Sections 3.3 & 3.4

1 Step response

Usually when putting performance specifications on the system, we look at the transfer function from the reference to the output:

$$\frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)}$$

The most common input used to specify performance is the *unit step*. In most cases, it is much easier in practice to produce something resembling a step change in the reference than it is to produce an impulse.

For the car example, we have:

$$\frac{X_f(s)}{R(s)} = \frac{\frac{K}{ms^2+bs}}{1 + \frac{K}{ms^2+bs}} = \frac{K}{ms^2 + bs + K}$$

Does a step change in $r(t)$ make any sense? This would be a step change in the position of the lead car which is not physically possible. However, a step change in velocity is more reasonable:

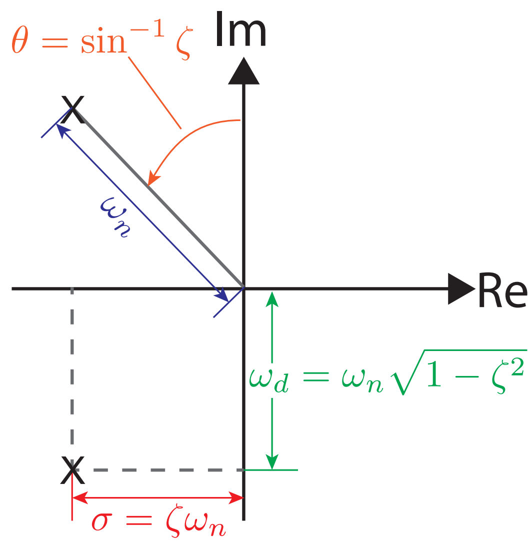
$$\frac{X_f(s)}{R(s)} = \frac{sX_f}{sR} = \frac{K}{ms^2+bs+K} \text{ (note that } \mathcal{L}^{-1}\left[\frac{sX_f}{sR}\right] = \frac{\dot{x}_f}{\dot{r}}\text{)}$$

So we can think of the step response as also describing the change in speed of the following vehicle if the lead vehicle speed suddenly changes.

This transfer function has a particular form that makes it easy to specify the step response. More complicated systems will often resemble this response even if they don't fit it exactly, if they have a dominant set of closed-loop poles (a single pair close to the imaginary axis).

We can consider the response of an underdamped, second-order system in terms of a natural frequency, ω_n, and a damping ratio, ζ.

$$\frac{X_f(s)}{R(s)} = \frac{K}{ms^2 + bs + K} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$



$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\zeta = \frac{b}{2\sqrt{mK}}$$

$$\sigma = \frac{b}{2m} = \zeta\omega_n$$

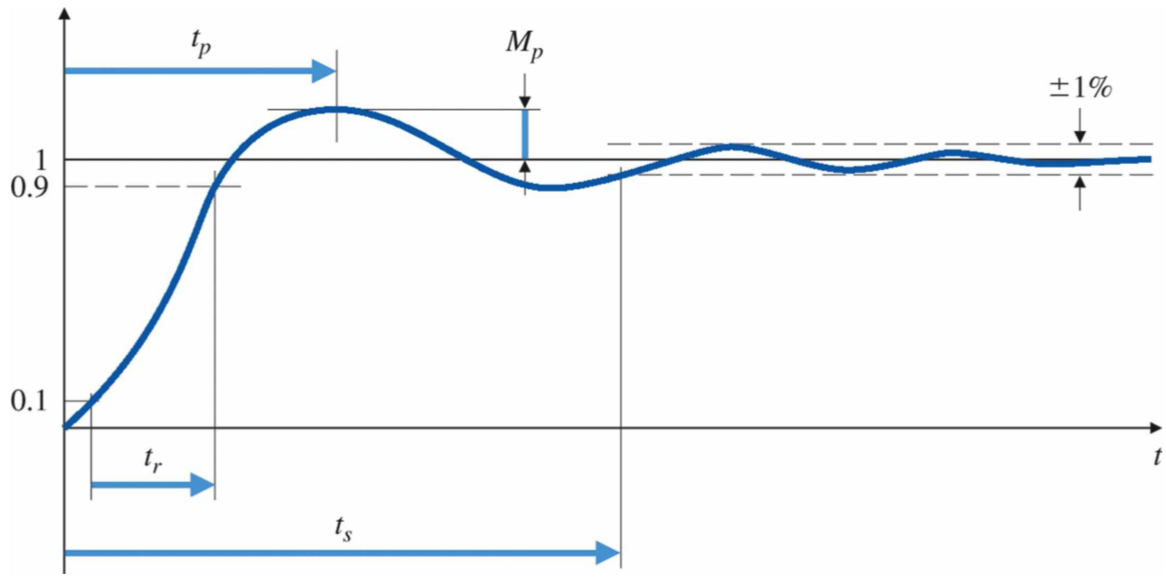
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

For a step response we have:

$$Y(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \cdot \frac{1}{s}$$

$$\Rightarrow y(t) = 1 - e^{-\sigma t}\left[\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right]$$

In general this will yield the following type of response:



See Section 3.3 for more examples with different damping ratios

2 Performance criteria

For the response above, we can create a number of performance specifications:

Rise time

Time until the response reaches the vicinity of its new steady state value (approximately 90%). This is based on observation and is not exact.

$$t_r \approx \frac{1.8}{\omega_n}$$

Peak time

Time until the response (if it is stable) reaches its maximum value. This occurs at the top of the peak which corresponds to a zero slope. We can derive an exact solution for the peak time, t_p.

$$\dot{y}(t) = \sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) - e^{-\sigma t}\left(-\omega_d \sin\omega_d t + \sigma\cos\omega_d t\right)$$

$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}\sin\omega_d t + \omega_d \sin\omega_d t\right) = 0$$

$$\Rightarrow \sin \omega_d t = 0 \text{ at the peak}$$

$$\text{thus, } t_p = \frac{\pi}{\omega_d}$$

Peak overshoot

This is maximum overshoot of the system, which occurs at $t = t_p$.

$$y(t) = 1 - \underbrace{e^{-\sigma t}}_{\text{decay}} \underbrace{\left(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right)}_{\text{oscillate}}$$

$$y(t_p) = 1 - e^{\frac{-\sigma \pi}{\omega_d}} \left(\cancelto{-1}{\cos \pi} + \frac{\sigma}{\omega_d} \cancelto{0}{\sin \pi} \right)$$

$$= 1 + e^{\frac{-\zeta \omega_n \pi}{\omega_n \sqrt{1-\zeta^2}}}$$

$$= 1 + \underbrace{e^{\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}}}_{M_p}$$

$$M_p = e^{\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}}$$

If $\zeta = 0.5, \quad M_p = 0.16$
$\zeta = 0.7, \quad M_p = 0.05$

Settling time (1%)

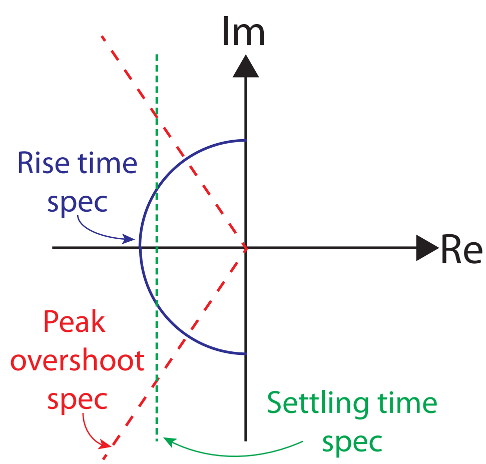
The time at which the response enters a band that is within 1% of the final value. Settling times can be similarly defined at other percentages of the final value.

$$y(t) = 1 - \underbrace{e^{-\sigma t}}_{\text{decay}} \underbrace{\left(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right)}_{\text{oscillate}}$$

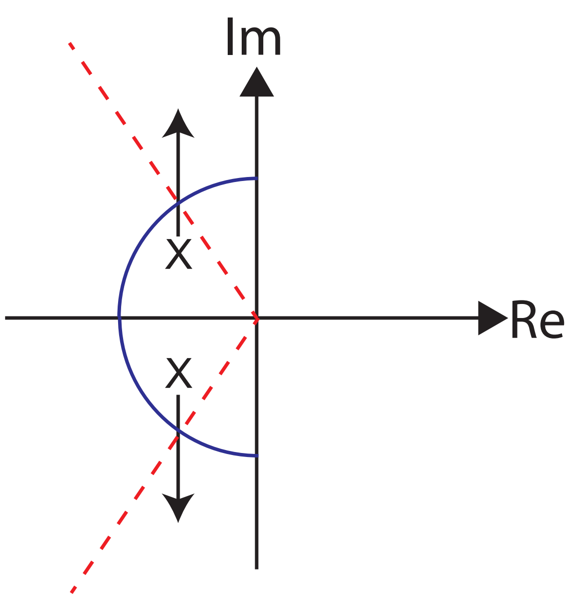
When $e^{-\sigma t} = 0.01$, the response must be within 1% of the final value of 1

$$\sigma t_s = -\ln(0.01) = 4.6 \quad \Rightarrow \quad t_s = \frac{4.6}{\sigma}$$

These have nice graphical representation in the s-plane:



Returning to our car-following example above, what can we do by changing the proportional gain?



We can make a damping spec
with a long rise time
-or-
we can make a rise time spec
with low damping.

The gain does not affect the
settling time.