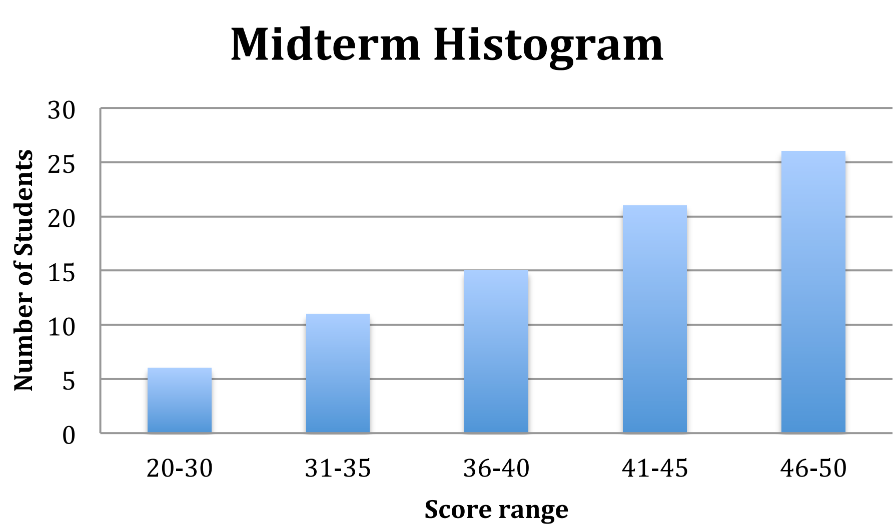
ENGR 105: Feedback Control Design
Midterm Exam, Winter Quarter 2012
Friday, Feb. 17, 2012, 9:00-9:50 am, Room 420-40

Name: __SOLUTIONS____________

Problem	Mean	Std. Dev.
1 (5 pts.)	3.5	1.1
2 (15 pts.)	14.6	0.8
3 (15 pts.)	12.9	2.9
4 (15 pts.)	10.2	4.4
Total (50 pts.)	41.2	6.6



Problem 1. (5 pts.)

Mark the following True (T) or False (F):

T a. (1 pt.) The time-domain performance specification of maximum overshoot that we defined in class for a system's response to a step input is only exact when the system transfer function can be written in the form:

$$\frac{a}{s^2 + bs + c}$$

(needs to be a second-order system with no zeros)

where a, b, and c are real constants.

T b. (1 pt.) Exact derivative control is impossible to implement on a real system.

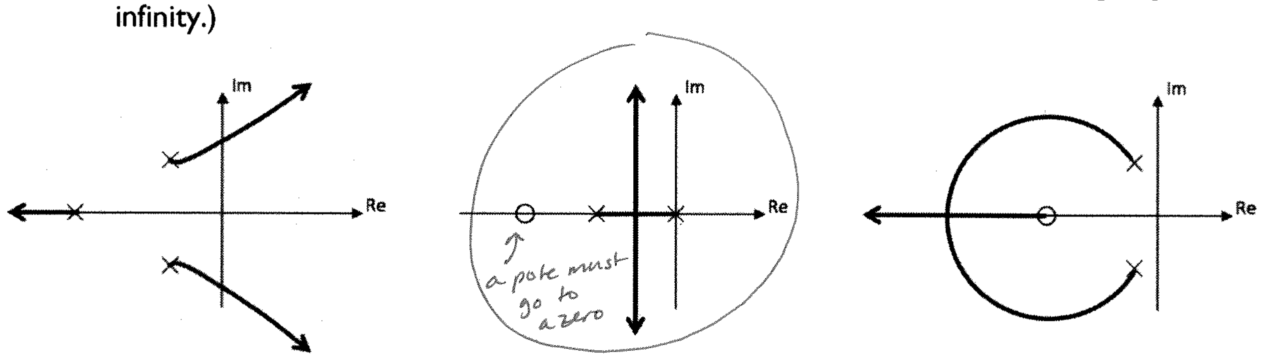
F c. (1 pt.) A system with the following transfer function is stable for a positive real number a.

$$\frac{1}{s^2 + as}$$

(this is marginally stable, which is not stable)

Circle the correct answer for the following questions:

d. (1 pt.) Which of the following cannot be a valid root locus? (Arrows indicate lines going to infinity.)



e. (1 pt.) Which of the following statements is *not* true?

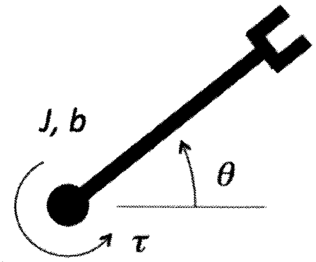
The Laplace transform is a nonlinear transformation. $\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$

The Laplace transform can be used to transform a system described by a differential equation into a system described by an algebraic equation.

Multiplication in the s-domain is equivalent to convolution in the time domain.

Problem 2. (15 pts.)

Consider the one-degree-of-freedom robot arm shown below. It has inertia J and torsional damping b. A motor (whose dynamics you can ignore) applies a torque τ at the center of rotation. The angle of the arm is given by θ.



a. (3 pts.) Develop the equation of motion for this plant.

$$\sum \tau = \tau - b\dot{\theta} = J\ddot{\theta}$$

$$\boxed{J\ddot{\theta} + b\dot{\theta} = \tau}$$

b. (3 pts.) Obtain the transfer function from torque input to angle output. Assume zero initial conditions.

$$\mathcal{L}\left[J\ddot{\theta} + b\dot{\theta}\right] = \mathcal{L}\left[\tau\right]$$

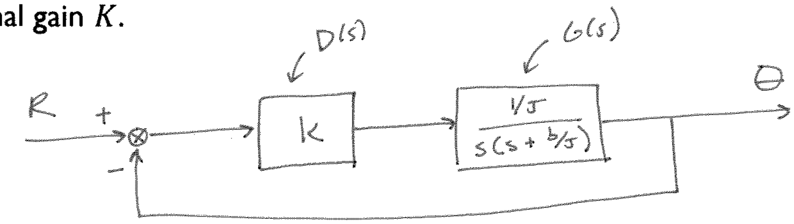
$$J\Theta(s)s^2 + b\Theta(s)s = T(s)$$

$$\Theta(s)\left[Js^2 + bs\right] = T(s)$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + bs} = \frac{1}{s(Js+b)}$$

$$\boxed{\frac{\Theta(s)}{T(s)} = \frac{1/J}{s(s + b/J)}}$$

c. (3 pts.) You want to design a feedback control system that will cause the arm angle to track a reference input. Draw the block diagram for the entire system, using unity feedback and a proportional gain K.



d. (3 pts.) Find the closed-loop transfer function from reference input to angle output for the closed-loop system.

$$\frac{\Theta(s)}{R(s)} = \frac{D(s)\,G(s)}{1 + D(s)\,G(s)} = \frac{K\,\frac{1}{s(Js+b)}}{1 + K\,\frac{1}{s(Js+b)}}$$

$$= \frac{K}{s(Js+b) + K}$$

$$\boxed{\frac{\Theta(s)}{R(s)} = \frac{K/J}{s^2 + b/J\, s + K/J}}$$

e. (3 pts.) For what values of K is this system stable?

characteristic equation: $s^2 + b/J\, s + K/J = 0$

$$Js^2 + bs + K = 0$$

roots: $\dfrac{-b \pm \sqrt{b^2 - 4JK}}{2J}$

roots can only have positive real part if

$$\sqrt{b^2 - 4JK} > b$$

we require $b^2 - 4JK < b^2$

$$\boxed{K > 0}$$

Problem 3. (15 pts.)

$U(s) = 1$

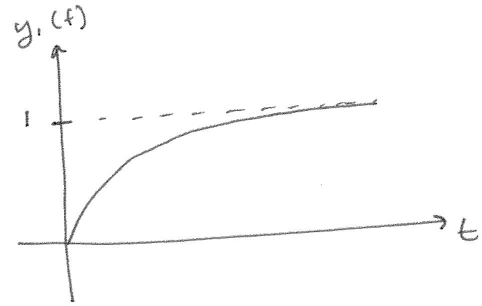
a. (5 pts.) Find the impulse response $y_1(t)$ for a system with a transfer function given below, and sketch the response versus time.

$$\frac{Y_1(s)}{U(s)} = \frac{1}{s(s+1)}$$

from table,

$y_1(t) = 1 - e^{-at}$ where $a = 1$

$$\boxed{y_1(t) = 1 - e^{-t}}$$



b. (5 pts.) Now consider the system below, which is the same except for the addition of a zero. Find the impulse response $y_2(t)$ for this new system and sketch is versus time.

$$\frac{Y_2(s)}{U(s)} = \frac{s+2}{s(s+1)}$$

$$Y_2(s) = \frac{A}{s} + \frac{B}{s+1}$$

$$= \frac{A(s+1) + Bs}{s(s+1)}$$

$$= \frac{s(A+B) + A}{s(s+1)}$$

$\therefore A = 2$

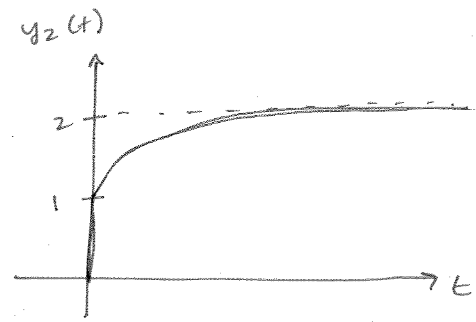
$A + B = 1$

$B = -1$

$$Y_2(s) = \frac{2}{s} + \frac{-1}{s+1}$$

from table, step

$$\boxed{y_2(t) = 2 \cdot 1(t) - e^{-t}}$$

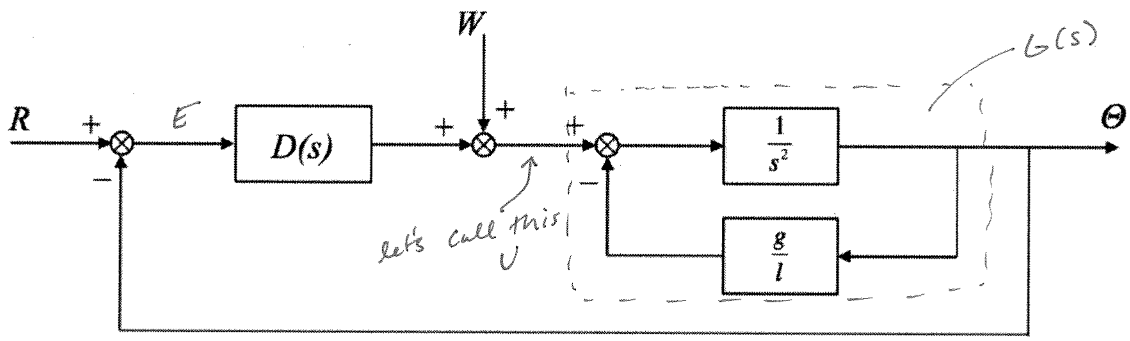


c. (5 pts.) Explain the effect of the addition of the zero.

- The zero instantly shifts the response up by 1 @ $t=0$.
- This is because the derivative in the numerator causes an instantaneous response at $t=0$.

Problem 4. (15 pts.)

Consider the system shown below, which represents control of the angle of a pendulum that has no damping. In this linearized model of a pendulum under the influence of gravity, g is the gravitational acceleration and l is the length of the pendulum. Both are positive real numbers.



a. (5 pts.) What condition must $D(s)$ satisfy so that the system is of Type 1 with respect to reference input?

First, find $G(s)$: $\left(U - \frac{g}{l}\Theta\right)\frac{1}{s^2} = \Theta$

$$U - \frac{g}{l}\Theta = \Theta s^2$$

$$\Theta\left(s^2 + g/l\right) = U$$

$$G(s) = \frac{\Theta}{U} = \frac{1}{s^2 + g/l}$$

For system type, look at error:

$$\frac{E}{R} = \frac{1}{1+DG} = \frac{1}{1 + D\left(\frac{1}{s^2 + g/l}\right)} = \frac{s^2 + g/l}{s^2 + g/l + D}$$

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{(s^2 + g/l)}{s^2 + g/l + D} \underbrace{\frac{1}{s^2}}_{} = \lim_{s\to 0} \frac{s^2 + g/l}{s^3 + g/l\, s + Ds}$$

R is a ramp for Type 1

as $s \to 0$, we want the quantity $\frac{s^2 + g/l}{s^3 + g/l\, s + Ds}$ to be a constant.

Thus, D needs to cancel out this s.

$\therefore$ $\boxed{\text{D must have a pole at } s = 0}$

(i.e. $D = \frac{1}{s}$ + other stuff)

b. (5 pts.) Let $D(s)$ be a PID controller. What PID controller gains will result in a stable closed-loop system?

$$D(s) = K_P + K_D s + K_I/s \qquad G = \frac{1}{s^2 + g/\ell} \qquad \text{let } g/\ell = a, \text{ so } G = \frac{1}{s^2+a}$$

Characteristic eqn is $1 + DG = 0$

$$1 + (K_P + K_D s + K_I/s)\left(\frac{1}{s^2+a}\right) = 0$$

$$s^2 + a + K_P + K_D s + K_I/s = 0$$

$$s^3 + K_D s^2 + (a + K_P)s + K_I = 0$$

Routh array:

row 3: $1 \quad a + K_P$

2: $K_D \quad K_I$

1: $b_1 \quad 0$ → $-K_I/K_D + a + K_P$

0: c_1 → K_I

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a+K_P \\ K_D & K_I \end{bmatrix}}{K_D} = \frac{-(K_I - K_D(a+K_P))}{K_D} = -K_I/K_D + a + K_P$$

$$c_1 = -\frac{\det\begin{bmatrix} K_D & K_I \\ -K_I/K_D + a + K_P & 0 \end{bmatrix}}{-K_I/K_D + a + K_P} = \frac{-(0 - K_I(K_I/K_D + a + \dots))}{-K_I/K_D + a + K_P} = K_I$$

$\therefore \boxed{\begin{matrix} K_D > 0 \\ K_I > 0 \end{matrix}}$

$-K_I/K_D + a + K_P > 0$

$\rightarrow \boxed{K_P > K_I/K_D - g/\ell}$ w/ PID controller & assume stable values of K_P, K_D, K_I

c. (5 pts.) Find the class of disturbances $w(t)$ that the system can reject with zero steady-state error.

$$\frac{E}{W} = \frac{-G}{1+DG} = \frac{\frac{-1}{s^2+a}}{1 + (K_P + K_D s + K_I/s)\left(\frac{1}{s^2+a}\right)} = \frac{-1}{s^2 + a + K_P + K_D s + K_I/s}$$

$$= \frac{-1}{s^2 + K_D s + (a + K_P) + K_I/s}$$

$$\lim_{s\to 0} sE(s) = \lim_{s \to 0} s \frac{-1}{s^2 + K_D s + (a+K_P) + K_I/s} \cdot W$$

this can equal zero if $W = \frac{a}{s}$ ← some constant

$\therefore \boxed{w(t) = 1(t)}$ ← a step (or a step multiplied by a constant)