ENGR 105: Feedback Control Design
Winter 2013

Lecture 11 - The Routh Array

Friday, February 1, 2013

Today's Objectives

1. define the Routh Array
2. apply the Routh Array to determine stability of a PI controller
3. examine the special case of a zero in the first column of the Routh Array

Reading: FPE Section 3.6

1 Define the Routh Array

The Routh Array (dating back to 1874) provides a way to analytically determine the stability of a system when the characteristic equation is of higher order. It also gives insight into the range of parameters for which a system is stable.

Consider a characteristic equation written in the form

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1}s + a_n = 0$$

Arrange the coefficients of the characteristic polynomial into two rows, beginning with the first and second coefficients and followed by the even- and odd-numbered coefficients. Add subsequent rows to form the *Routh Array*:

$$\begin{array}{llrcccc}
\text{Row} & n & s^n\!: & 1 & a_2 & a_4 & \dots \\
\text{Row} & n-1 & s^{n-1}\!: & a_1 & a_3 & a_5 & \dots \\
\text{Row} & n-2 & s^{n-2}\!: & b_1 & b_2 & b_3 & \dots \\
\text{Row} & n-3 & s^{n-3}\!: & c_1 & c_2 & c_3 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{Row} & 2 & s^2\!: & * & * & & \\
\text{Row} & 1 & s\!: & * & & & \\
\text{Row} & 0 & s^0\!: & * & & &
\end{array}$$

Where:

$$b_1 = \frac{-\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} \qquad b_2 = \frac{-\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$c_1 = \frac{-\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} \qquad c_2 = \frac{-\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

... and so on ...

All roots of the characteristic equation are negative if and only if all terms in the first column are greater than zero. The characteristic equation has as many right half plane roots as there are sign changes in the first column (for example, $++--$ is one sign change; $++-+$ is two sign changes).

A slight modification to this method is necessary when only the first column contains a zero (and therefore is neither strictly positive or negative).

2 Application to the PI controller (car-following example)

With the PI controller in the car following example, the system type was correct but the characteristic equation became third order. We can check its stability using the Routh Array:

$$ms^3 + bs^2 + K_p s + K_i = 0$$
$$\Rightarrow s^3 + \frac{b}{m}s^2 + \frac{K_p}{m}s + \frac{K_i}{m} = 0$$

We then form the Routh array:

Row 3:	1	K_p/m	0
Row 2:	b/m	K_i/m	0
Row 1:	b_1		
Row 0:	c_1		

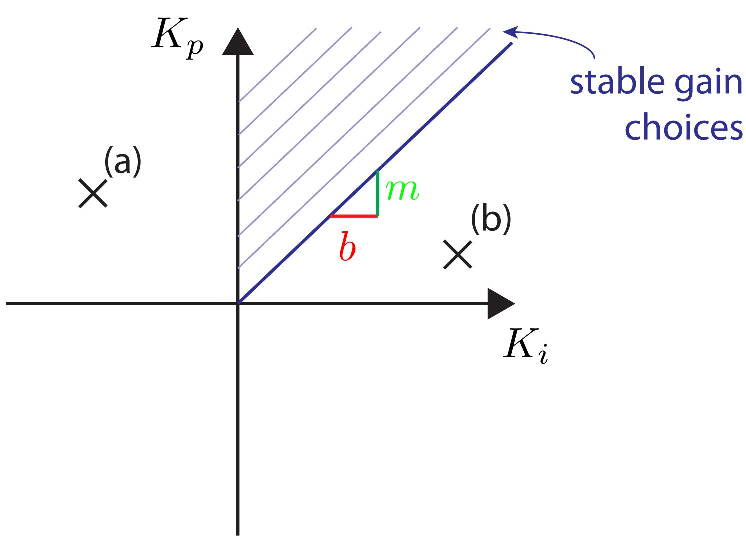
$$b_1 = \frac{-\det \begin{bmatrix} 1 & K_p/m \\ b/m & K_i/m \end{bmatrix}}{b/m} = \frac{-\left(K_i/m - bK_p/m^2\right)}{b/m} = K_p/m - K_i/b$$

$$c_1 = \frac{-\det \begin{bmatrix} b/m & K_i/m \\ \left(K_p/m - K_i/b\right) & 0 \end{bmatrix}}{\left(K_p/m - K_i/b\right)} = \frac{\left(K_p/m - K_i/b\right)\left(K_i/m\right)}{\left(K_p/m - K_i/b\right)} = K_i/m$$

The first column therefore looks like:

Row	3:	1	Always > 0
Row	2:	b/m	Aways > 0 since $b, m > 0$
Row	1:	$K_p/m - K_i/b$	$\Rightarrow K_p > K_i\,(m/b)$
Row	0:	K_i/m	$\Rightarrow K_i > 0$

We can envision the region of stable gains graphically:



What if we chose gains at point (a) or (b)?

Let's look at the first column of the Routh array for the two cases (a) and (b):

		Gains at (a)	Gains at (b)
Row	3:	1	1
Row	2:	$^{b}/_{m}$	$^{b}/_{m}$
Row	1:	$^{K_p}/_{m} - {}^{K_i}/_{b} > 0$	$^{K_p}/_{m} - {}^{K_i}/_{b} < 0$
Row	0:	$^{K_i}/_{m} < 0$	$^{K_i}/_{m} > 0$
		$\Downarrow$	$\Downarrow$
		1 sign change	2 sign changes
		1 unstable root	2 unstable roots

3 Modification for zero in the first column

If the first element in a row of the Routh Array is zero, the array requires modifications to avoid dividing by zero. The idea here is to replace the zero with a small positive constant $\epsilon > 0$, proceed as before, then apply the stability criterion by taking the limit as $\epsilon \to 0$.

For example, if our system has no damping, $b = 0$.

Row	3:	1	$^{K_p}/_{m}$	0
Row	2:	$0 \rightarrow$ replace with ϵ	$^{K_i}/_{m}$	0
Row	1:	$-{}^{K_i}/_{m\epsilon} + {}^{K_p}/_{m}$		
Row	0:	$^{K_i}/_{m}$		

For stability,

$$^{K_i}/_{m} > 0$$

$$\lim_{\epsilon \to 0} -{}^{K_i}/_{m\epsilon} + {}^{K_p}/_{m} > 0$$

$$\lim_{\epsilon \to 0} K_p > {}^{K_i}/_{\epsilon} \qquad \text{not possible!}$$

Without damping, the system will have two unstable roots for any positive K_p and K_i.

Further modification is required in the special case when an entire row of the Routh Array is zero (see the text for this process).

The proof of the Routh Array is rather involved (though a simpler proof appeared about 125 years after the original). It shouldn't be too surprising, however, that the coefficients of the characteristic equation give information about the sign of the roots.

Back to the example, with PI control, the transfer function becomes

$$\frac{Y(s)}{R(s)} = \frac{X_f(s)}{R(s)} = \frac{K_p s + K_i}{ms^3 + bs^2 + K_p s + K_i}$$

$$= \frac{\frac{K_p}{m}s + \frac{K_i}{m}}{s^3 + \underbrace{\frac{b}{m}}_{\substack{\text{set by}\\ \text{system}}} s^2 + \underbrace{\frac{K_p}{m}s + \frac{K_i}{m}}_{\substack{\text{we can}\\ \text{choose}}}}$$

We can choose K_p and K_i to guarantee stability if $b > 0$, but we cannot place all three poles where we want them with only two control gains.