Assignment 4 Solutions

Problem 1

a) Use Kirchhoff's voltage law

$$\boxed{V_1(t) - L\frac{d}{dt}i(t) - R\,i(t) - \frac{1}{C}\int i(t)\,dt = 0}$$

b) $\boxed{i(t) = C\frac{d}{dt}V_2(t)}$

c) ① $V_1 - LsI - RI - \frac{1}{Cs}I = 0$

② $I = CsV_2$

plug ② into ①

$$V_1 - Ls(CsV_2) - R(CsV_2) - \frac{1}{Cs}CsV_2 = 0$$

$$V_1 = (LCs^2 + RCs + 1)V_2$$

$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}}$$

d) $\boxed{\omega_n = \frac{1}{\sqrt{LC}}}$

$$2\xi\omega_n = \frac{R}{L}$$

$$2\xi\frac{1}{\sqrt{LC}} = \frac{R}{L}$$

$$\boxed{\xi = \frac{R}{2\sqrt{L/C}}}$$

e) $.25 = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)$

$\xi \geq .404$

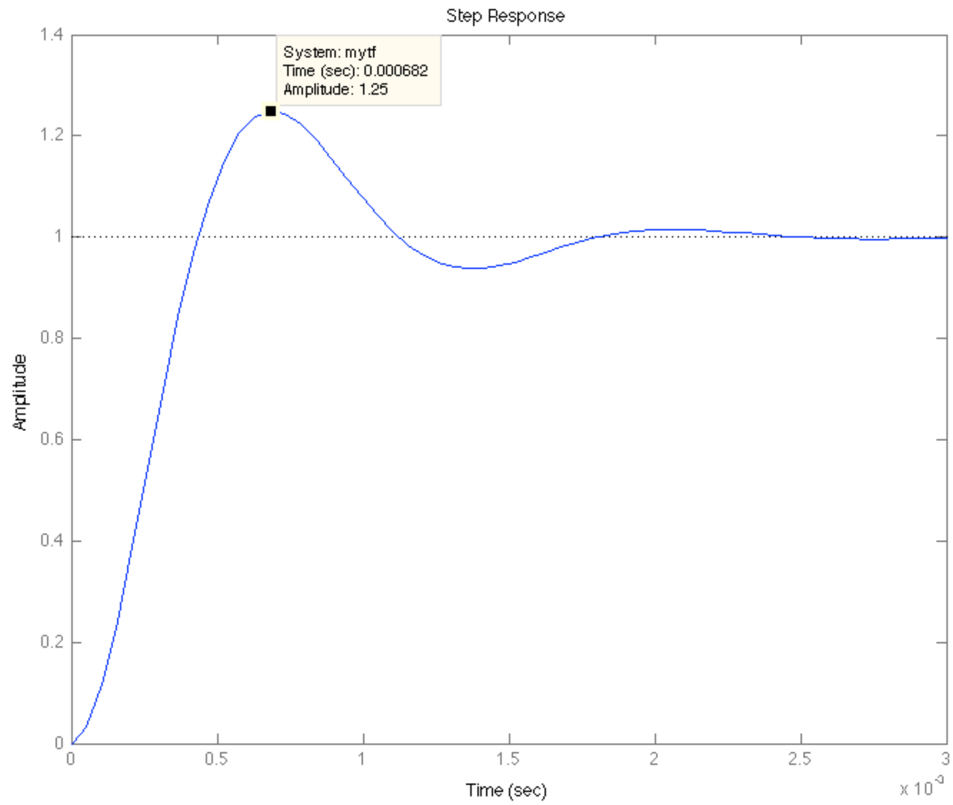
$$\frac{R}{2\sqrt{L/C}} \geq .404$$

$\boxed{R \geq 40.4\ \Omega}$

f) see next page for plot

from plot $\boxed{t_p \approx .68\text{ ms}}$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\left(\sqrt{\frac{1}{LC}}\right)\left(\sqrt{1-.404^2}\right)} = \boxed{.687\text{ ms}}$$




```
% parameters
R = 40.4;
C = 4e-6;
L = 10e-3;

% natural frequency and damping ratio
wn = 1/sqrt(L*C);
zeta = R/(2*sqrt(L/C))

% time to peak
tp = pi/(wn*sqrt(1-zeta^2))

% plot
mytf = tf(1,[L*C R*C 1]);
step(mytf)
set(gcf,'color','white')
```

Problem 2

a) closed-Loop TF

$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s^2+2s}}{1+\frac{K}{s^2+2s}} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

for 10% OS $\quad \zeta = .591$

$\omega_n = \sqrt{K}$

$2\zeta\omega_n = 2$

$(2)(.591)(\sqrt{K}) = 2$

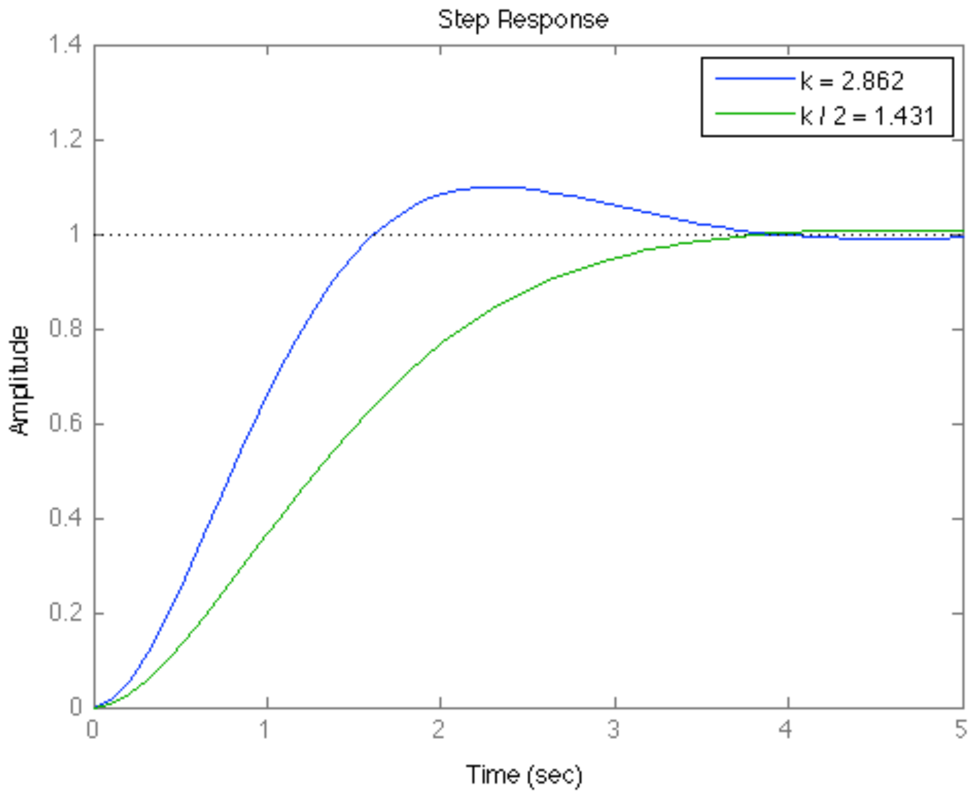
$\boxed{K = 2.86}$

b) $\zeta = \frac{1}{\sqrt{K}}$

as K increases, ζ decreases, $\boxed{\text{overshoot increases}}$

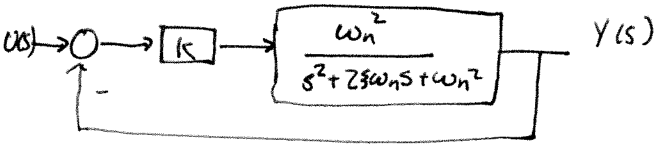
c) see next page

d) if overshoot would cause damage (for example, on a CNC mill), then the $K/2$ response is clearly preferable. If peak time were the most important parameter to minimize, then the K response is preferable.




```
k1 = 2.862;
tf1 = tf(k1,[1 2 k1]);
step(tf1);

hold on

k2 = k1/2;
tf2 = tf(k2,[1 2 k2]);
step(tf2);

legend('k = 2.862','k / 2 = 1.431')
set(gcf,'color','white')
```

Problem 5



Simplify block diagram

$$G(s) = \frac{K \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}{1+K\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}} = \frac{K\omega_n^2}{s^2+2\xi\omega_n s+(1+K)\omega_n^2}$$

new natural frequency and damping ratio of closed loop system:

$$\boxed{\omega_n' = \sqrt{(1+K)}\,\omega_n}$$

$$\cancel{2}\xi'\omega_n' = \cancel{2}\xi\omega_n$$

$$\xi' = \frac{\xi\omega_n}{\omega_n'} = \frac{\xi\cancel{\omega_n}}{\sqrt{1+K}\,\cancel{\omega_n}} = \frac{\xi}{\sqrt{1+K}}$$

$$\boxed{\xi' = \frac{\xi}{\sqrt{1+K}}}$$

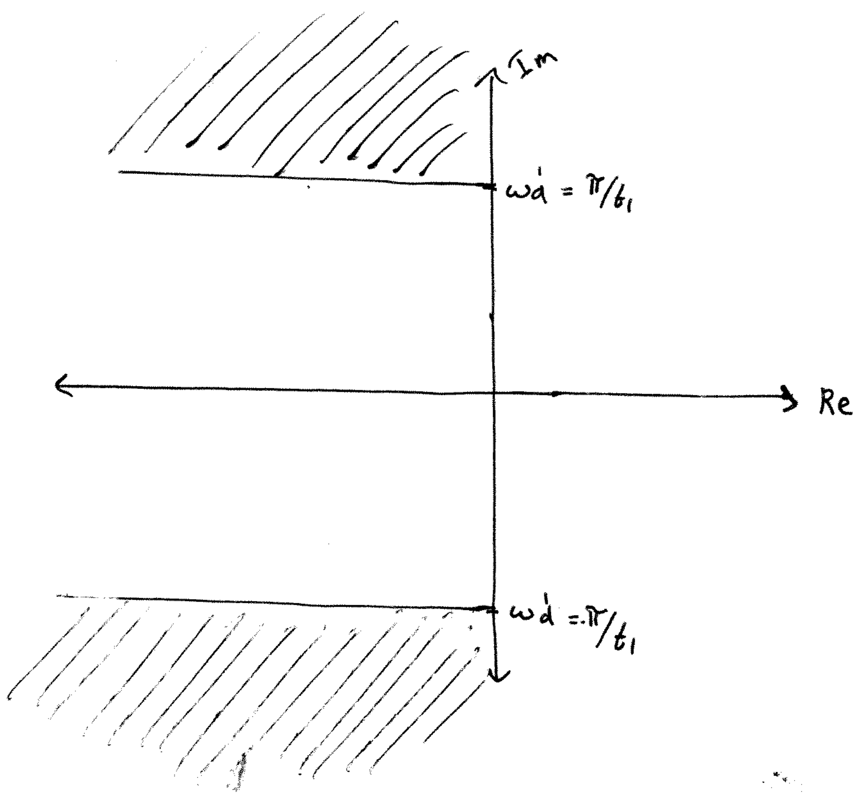
new damped natural frequency

$$\omega_d' = \omega_n'\sqrt{1-\xi^2}$$

$$t_p < \frac{\pi}{\omega_d}$$

$$\boxed{\omega_d' > \frac{\pi}{t_1}}$$

acceptable regions shaded



to be stable, poles must be in Left half plane

$$\omega_d' > \frac{\pi}{t_1}$$

$$\omega_n' \sqrt{1-(\zeta')^2} > \frac{\pi}{t_1}$$

$$\sqrt{1+K}\,\omega_n \sqrt{1-\left(\frac{\zeta}{\sqrt{1+K}}\right)^2} > \frac{\pi}{t_1}$$

$$(1+K)\omega_n^2 \left(1 - \frac{\zeta^2}{1+K}\right) > \left(\frac{\pi}{t_1}\right)^2$$

solve for K

$$\boxed{K > \zeta^2 + \left(\frac{\pi^2}{t_1^2 \omega_n^2}\right) - 1}$$

Problem 7

a) $J\theta s^2 + B\theta s = T_c$

$$\frac{\theta}{T_c} = \frac{1}{Js^2+Bs} = \frac{1/J}{s^2+\frac{B}{J}s}$$

$$\boxed{\frac{\theta}{T_c} = \frac{1.667\cdot10^{-6}}{s^2+\frac{1}{30}s}}$$

b) $J\theta s^2 + B\theta s = K(\theta_r - \theta)$

$$(Js^2+Bs+K)\theta = K\theta_r$$

$$\frac{\theta}{\theta_r} = \frac{K/J}{s^2+\frac{B}{J}K+\frac{K}{J}}$$

$$\boxed{\frac{\theta}{\theta_r} = \frac{1.667\cdot10^{-6}K}{s^2+\left(\frac{1}{30}\right)s+1.667\cdot10^{-6}K}}$$

c) $.1 = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)$

$\xi = .591$

$(2)(.591)(\omega_n) = \frac{1}{30}$

$\omega_n = .0282 = \sqrt{1.667\cdot10^{-6}K}$

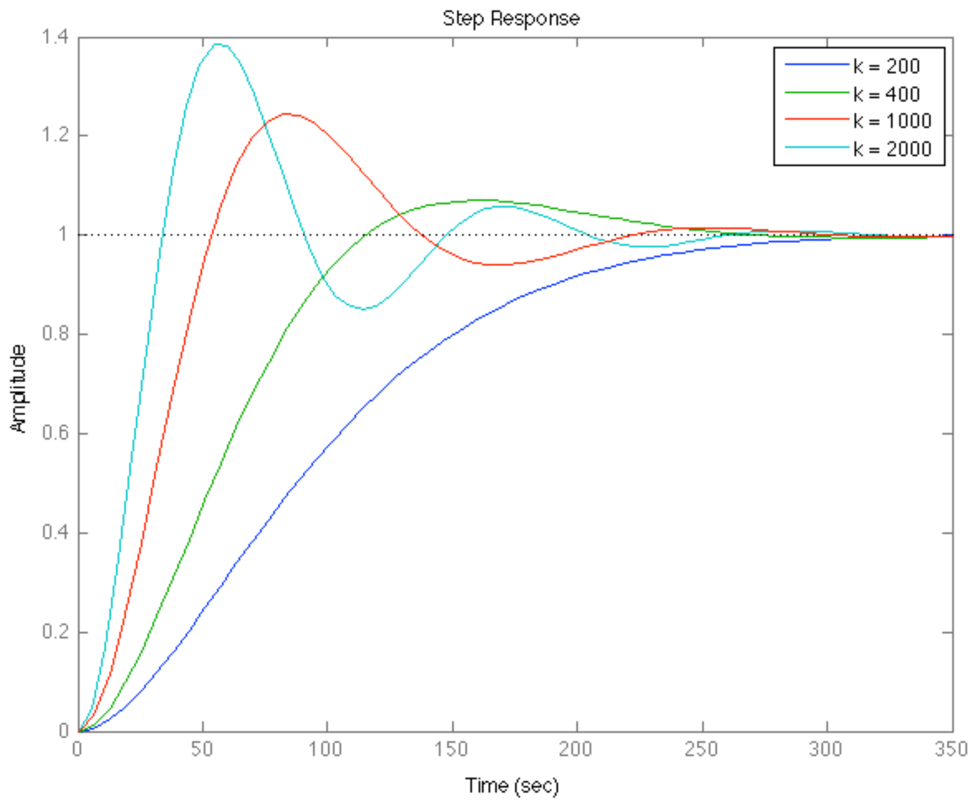
$\boxed{K = 477}$

d) $t_r = \frac{1.8}{\omega_n}$

$$80 > \frac{1.8}{\sqrt{1.667\cdot10^{-6}K}}$$

$\boxed{K > 304}$

e) see next page



Overshoot and rise times (from the stepinfo() command)

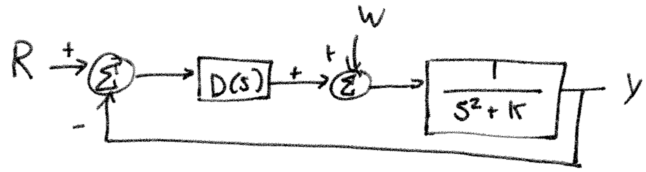
K	OS	tr [s]
200	.1%	161
400	7.0%	76.3
1000	24.5%	36.5
2000	38.4%	23.1

The plots confirm the calculations from part (d).

```
k = 200;
tf1 = tf(1.667e-6*k,[1 1/30 1.667e-6*k]);
step(tf1);
hold on

k = 400;
tf2 = tf(1.667e-6*k,[1 1/30 1.667e-6*k]);
step(tf2);

k = 1000;
tf3 = tf(1.667e-6*k,[1 1/30 1.667e-6*k]);
step(tf3);

k = 2000;
tf4 = tf(1.667e-6*k,[1 1/30 1.667e-6*k]);
step(tf4);

legend('k = 200','k = 400','k = 1000','k = 2000')
set(gcf,'color','white')
```

Problem 5

Simplify block diagram



a) $\dfrac{Y(s)}{R(s)} = \dfrac{\dfrac{D(s)}{s^2+K}}{1+\dfrac{D(s)}{s^2+K}} = \dfrac{D(s)}{s^2+K+D(s)}$

To track a ramp with constant steady-state error we need system Type 1

[redacted] The plant has no free integrators (Type 0), so the free integrator must be contributed by D(s)

$\boxed{D(s) \text{ must have a pole at } s=0}$

b) $\dfrac{Y(s)}{W(s)} = \dfrac{\dfrac{1}{s^2+K}}{1+\dfrac{D(s)}{s^2+K}} = \dfrac{1}{s^2+D(s)+K}$

$Y(s) = W(s) \cdot \dfrac{1}{s^2+D(s)+K}$

Disturbance class W(s) has form $\dfrac{1}{s^\ell}$

final value of Y(s)

$Y_{ss} = \lim_{s \to 0} s \cdot \dfrac{1}{s^\ell} \cdot \dfrac{1}{s^2+D(s)+K} = 0$ (because we want 0 ss error)

$= \lim_{s \to 0} \dfrac{1}{s^{\ell-1}} \cdot \dfrac{1}{s^2+D(s)+K} = 0$

this holds if and only if

$\lim_{s \to 0} s^{\ell-1} D(s) = \infty$

D(s) has one pole at the origin $\Rightarrow$ ~~[illegible]~~ a step can be rejected, but not a ramp

$\boxed{\text{The system can reject a step disturbance with 0 steady state error}}$