ENGR 105: Feedback Control Design
Winter 2013

Lecture 12 - PID and Derivative Control

Monday, February 4, 2013

Today's Objectives

1. introduce the PID controller and its effect on stability and system type
2. show how derivative (actually, PD) control affects system behavior
3. explain practical implementation issues associated with derivative control
4. present the lead compensator, an approximation of derivative control
5. show the effect of a zero (s in the transfer function numerator) on system behavior

Reading: FPE Sections 4.3

1 PID Controller

For the third control knob we can add the derivative of the error to the control signal:

$$U(s) = K_p E(s) + K_d s E(s) + \frac{K_i}{s} E(s)$$

$$D(s) = \frac{U(s)}{E(s)} = K_p + K_d s + \frac{K_i}{s}$$

$$u(t) = \underset{\text{“spring”}}{\underset{\uparrow}{K_p e(t)}} + \underset{\text{“damper”}}{\underset{\uparrow}{K_d \dot{e}(t)}} + K_i \int_0^t e(t)\ \mathrm{d}t$$

This is easy enough to implement when there is a direct measurement of $\dot{e}(t)$ available. In the car-following example, radar usually gives both range and range rate, so this measurement exists. However, it can be tricky to get $\dot{e}(t)$ from measurements of $e(t)$ alone – this will be discussed in more detail later.

Let's look at our example with the complete PID controller:

$$G(s) = \frac{1}{s\,(ms + b)} \qquad\qquad D(s) = \frac{K_p s + K_d s^2 + K_i}{s}$$

$$\frac{Y(s)}{R(s)} = \frac{DG}{1+DG} = \frac{K_d s^2 + K_p s + K_i}{ms^3 + \underbrace{(b+K_d)}_{\substack{\text{derivative gain}\\ \text{adds to damping}\\ \text{and adds}\\ \text{another zero}}} s^2 + K_p s + K_i}$$

Checking stability with the Routh Array: $\quad s^3 + \frac{(b+K_d)}{m}s^2 + \frac{K_p}{m}s + \frac{K_i}{m} = 0$

Row 3:	1	${K_p}/{m}$	0
Row 2:	$\frac{(b+K_d)}{m}$	${K_i}/{m}$	
Row 1:	${K_p}/{m} - {K_i}/{(b+K_d)}$		
Row 0:	${K_i}/{m}$		

So the conditions for stability are:

$$b + K_d > 0$$

$$K_i > 0$$

$$K_p > \frac{m}{b+K_d}K_i$$

Checking system type:

$$\frac{E(s)}{R(s)} = \frac{1}{1+DG} = \frac{s^2(ms+b)}{ms^3 + (b+K_d)s^2 + K_p s + K_i} \qquad \text{Type 2}$$

$$\frac{E(s)}{W(s)} = \frac{-G}{1+DG} = \frac{s}{ms^3 + (b+K_d)s^2 + K_p s + K_i} \qquad \text{Type 1}$$

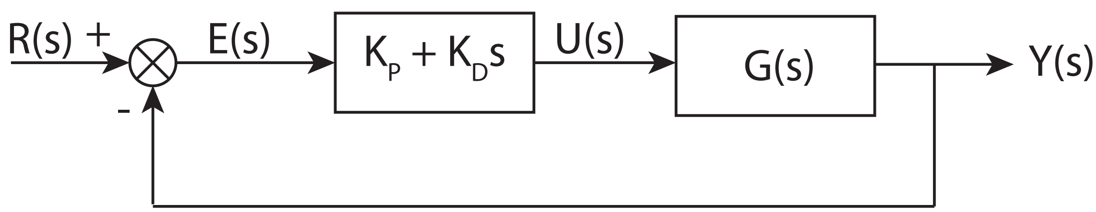
The addition of derivative control did not change the system type at all.

PID is often used because the control gains have (at least roughly) intuitive meanings and can often be tuned by hand:

Speed of response $\rightarrow K_p$
Damping or overshoot $\rightarrow K_d$
Steady-state error rejection $\rightarrow K_i$

2 Derivative Control

When the steady-state error response does not require integral control, a PD controller may be used instead.



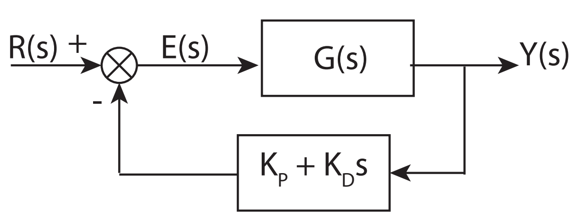
$$\frac{Y(s)}{R(s)} = \frac{(K_p + K_d s)\, G(s)}{1 + (K_p + K_d s)\, G(s)}$$

For the transfer function $G(s) = \dfrac{1}{ms^2 + bs}$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{K_p + K_d s}{ms^2 + (b + K_d)\, s + K_p}$$

The denominator looks like a familiar form (now the system is second order since there is no integral control), but the system has a zero.

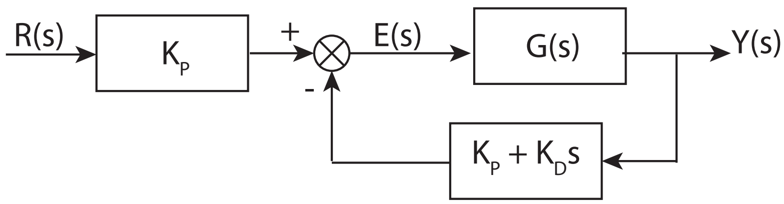
What happens if we put the controller in the feedback loop instead of the forward path?



$$\begin{aligned}
&E = R - Y\,(K_p + K_d s) \\
&Y = E\,G \rightarrow E = \frac{Y}{G} \\
&\frac{Y}{G} = R - Y\,(K_p + K_d s) \\
&Y = GR - YG\,(K_p + K_d s) \\
&GR = Y\,(1 + G\,(K_p + K_d s))
\end{aligned}$$

$$\frac{Y(s)}{R(s)} = \frac{G}{1 + G\,(K_p + K_d s)} = \frac{1}{ms^2 + (b + K_d)s + K_p}$$

To deal with the steady-state error, we can scale the reference. This has no effect on the stability of the control system since it occurs outside the feedback loop:



$$\frac{Y(s)}{R(s)} = \frac{K_p}{ms^2 + (b + K_d)\,s + K_p} \qquad \rightarrow \qquad \text{steady state gain} = 1$$

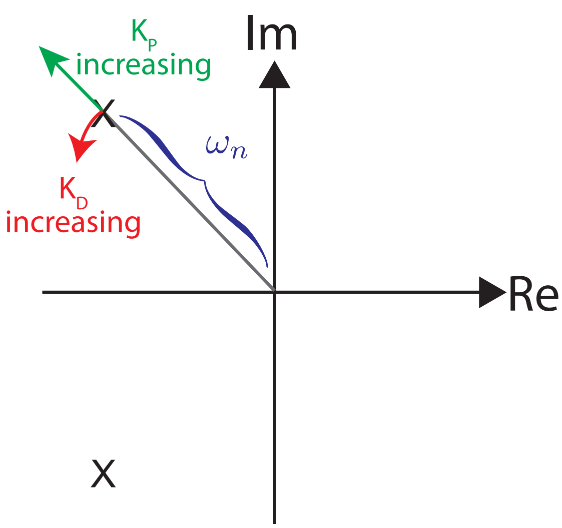
If we equate the terms to those in the standard form:

$$\frac{K_p}{ms^2 + (b + K_d)\,s + K_p} = \frac{K_p/m}{s^2 + \frac{(b+K_d)}{m}s + K_p/m} = \frac{{\omega_n}^2}{\underbrace{s^2 + 2\zeta\omega_n s + {\omega_n}^2}_{\substack{\text{scaling by a constant} \\ \text{just scales the output}}}}$$

K_p can be used to set the natural frequency

K_d can be used to set the damping ratio

The behavior of the closed-loop poles as we change gains can be viewed graphically:



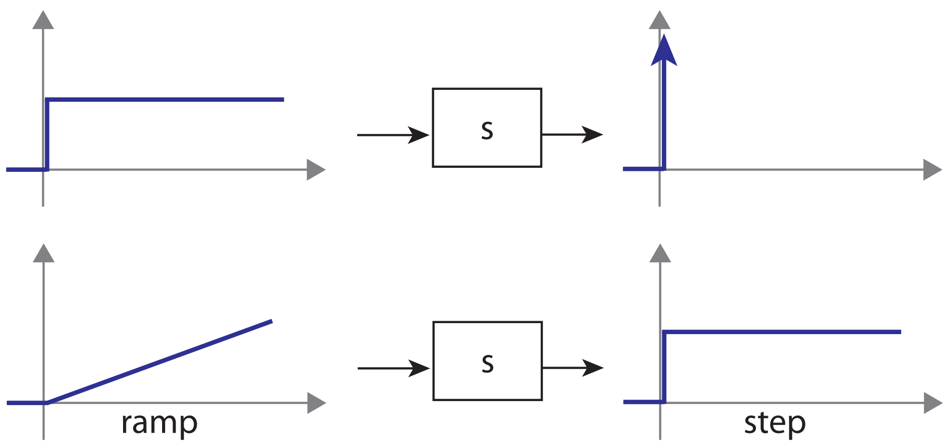
We can use these gains to achieve specifications like rise time and peak overshoot simultaneously.

3 Practical Issues with Derivative Control

Derivative control can be a powerful tool in achieving the desired performance of a system. However, there is a challenge that pure derivative control is not actually realizable!

$$D(s) = \frac{K_p + K_d s}{1} \qquad \Rightarrow \qquad \text{numerator has higher order than the denominator}$$

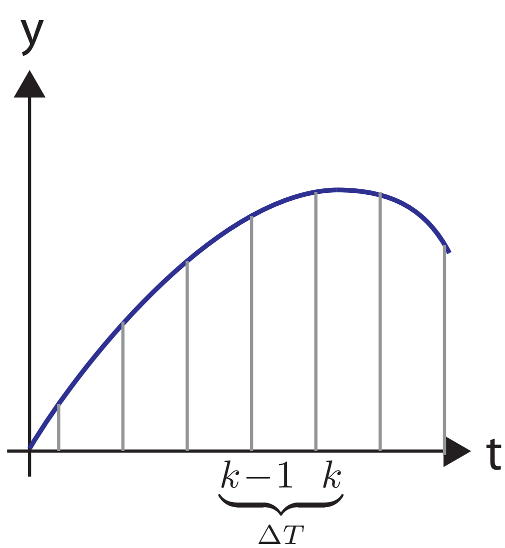
Systems with a higher numerator order than denominator order cannot be built with passive elements like resistors, capacitors, and inductors. They also have issue with causality (outputs depend on things that haven't happened yet) and power. Consider a couple of example inputs:



Finite input gives an infinite output

Output changes before the input does

In digital systems, we might think of approximating a derivative with a difference.



$$\dot{y}(k) \approx \frac{y(k) - y(k-1)}{\Delta T}$$

This is inherently looking backwards and not the true derivative.

This approach often results in a very noisy signal since the differencing process amplifies noise.

Suppose $\quad y_m(k) = y(k) + n(k)$

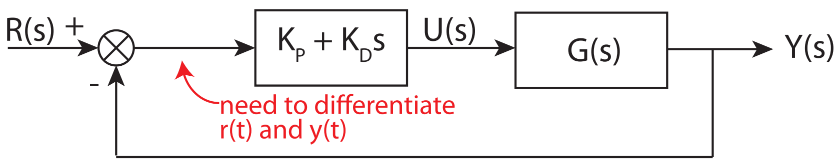
Then $\quad \dot{y}_m(k) = \frac{y(k)-y(k-1)}{\Delta T} + \frac{n(k)-n(k-1)}{\Delta T}$

y_m - measured output
y - actual output
n - noise

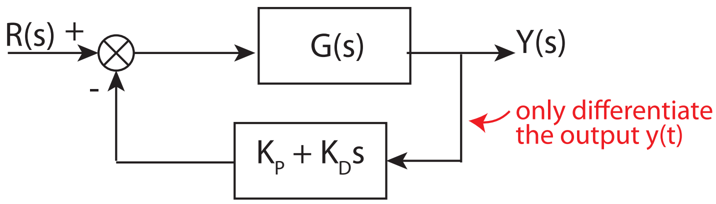
As $\Delta T \to 0$, the approximation of the derivative gets better, but the noise amplification gets worse.

With very clean signals (like position signals from a high-resolution optical encoder) this may work to approximate the derivative. In other cases, it can result in big problems.

The challenge of physically implementing a derivative is also a deciding factor in considering whether to put the controller in the feedback or forward path:



$$\begin{aligned} u(t) =& K_p\left(r(t) - y(t)\right) \\ &+ K_d\left(\dot{r}(t) - \dot{y}(t)\right) \end{aligned}$$



$$\begin{aligned} u(t) =& r(t) - K_p y(t) \\ &- K_d \dot{y}(t) \end{aligned}$$

If the reference signal is noisy, it may be preferable to have the controller in the feedback path to avoid differentiating it. The feedback path is also better if the reference input makes step changes. (What will the input $u(t)$ try to do in such cases?).

4 The Lead Compensator

The issues with implementing pure derivative control can be resolved by using a lead compensator. this is a circuit that can be built (remember the first homework assignment!) and filters out the high frequency noise (this will be clearer in a couple of weeks).

The lead compensator is just the PD controller with a fast pole:

$$D(s) = \frac{K_p + K_d s}{\frac{1}{a}s + 1} \qquad \text{pole at } s = -a$$

As the value of a increases, the lead compensator resembles the PD controller more closely.

5 The Effect of a Zero

Another difference between choosing to put the PD controller in the feedback path or the forward path is the zero.

$$\underbrace{\frac{Y(s)}{R(s)} = \frac{K_p + K_d s}{ms^2 + (b + K_d)\,s + K_p}}_{\text{forward path}} \qquad \underbrace{\frac{Y(s)}{R(s)} = \frac{K_p}{ms^2 + (b + K_d)\,s + K_p}}_{\text{feedback path}}$$

What is the effect of this zero? There are several ways to look at zeros and this will be the focus of the next lecture. For now, consider breaking up the two parts of the numerator for the "forward path" controller.

$$\frac{Y(s)}{R(s)} = \underbrace{\frac{K_p}{ms^2 + (b + K_d)\,s + K_p}}_{\substack{\text{our standard 2}^{\text{nd}}\\ \text{order system form}}} + s\underbrace{\frac{K_d}{ms^2 + (b + K_d)\,s + K_p}}_{\substack{\text{the derivative of the}\\ \text{standard 2}^{\text{nd}}\text{ order form}}}$$

If $K_d << K_p$, the system response looks a lot like the standard second order system. In this case, the zero location is far from the imaginary axis since

$$z = \frac{K_p}{K_d} \text{ is the zero location}$$

Therefore, zeros that are far from the imaginary axis have little impact on the system behavior *when they are in the left half plane.*

As K_d becomes larger, however, the system response includes more of this derivative term. This can lead to, for instance, greater overshoot to a step response.