ENGR 105: Feedback Control Design
Winter 2013

# Lecture 15 - Root Locus Analysis

Monday, February 11, 2013

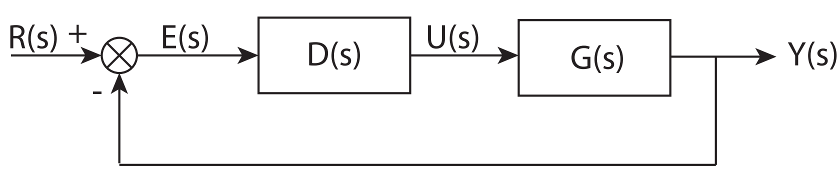
### Today's Objectives

1. definition of the root locus
2. relationship between root locus and phase
3. the first two rules for drawing the root locus

Reading: FPE Sections 5.1, 5.2

## 1 Root Locus

Zeros in an open-loop transfer function influence the closed-loop poles when a feedback loop is closed:




This system is stable if the roots of $1 + D(s)G(s) = 0$ have negative real parts.

We can rewrite this in terms of a control gain $K$ and the open loop poles and zeros,

$$1 + K\frac{b(s)}{a(s)} = 0$$

$$\Rightarrow a(s) + Kb(s) = 0$$

$\uparrow$ open loop poles  $\uparrow$ open loop zeros

solutions to this equation are the closed-loop poles

If we have $D(s) = K$, then $a(s)$ and $b(s)$ are just the poles and zeros of $G(s)$. If the controller is more complicated and has dynamics itself, then $a(s)$ and $b(s)$ will be a combination of plant and controller.

The key idea behind a root locus analysis is to be able to see how pole locations vary with the system gain. In our simple proportional control of the car follower,

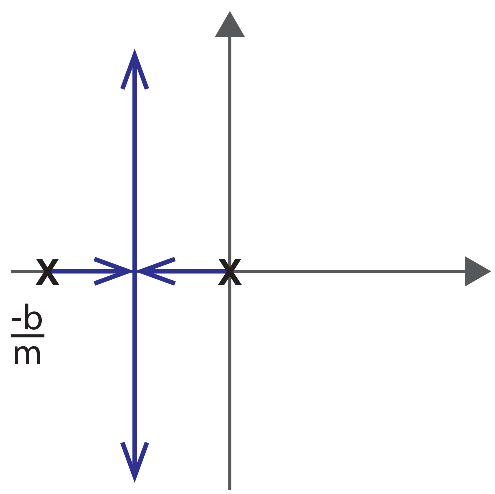
$$D(s) = K \qquad G(s) = \frac{1}{ms^2 + bs}$$

$$\text{Characteristic equation: } ms^2 + bs + K = 0$$

How do the roots vary as the gain changes?

$$s = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

when $K = 0$,
poles are at $s = 0, -\frac{b}{m}$




As $K$ is increased, the open loop poles move toward one another before breaking away from the real axis and moving toward infinity.

This is known as a *root locus* and shows us all possible locations of the closed-loop poles.

The root locus was developed by Walter R Evans (1948) specifically for investigating aircraft dynamic responses. Although developed before computers, the root locus is still useful now because:

- The root locus rules provide a lot of intuition about closed-loop system behavior.
- MATLAB can generate a root locus very quickly, giving a good graphical description of the changing system response

In order to perform root locus analysis, we first need to arrange our characteristic equation to fit the following basic form:

$$a(s) + Kb(s) = 0 \qquad \text{or} \qquad 1 + K\frac{b(s)}{a(s)} = 0$$

Where $a(s)$ and $b(s)$ are monic (the highest power of $s$ has a coefficient of 1). If our system has several gains, we can do this for each one separately.

For example, if

$$\frac{Y(s)}{R(s)} = \frac{K_d s + K_p}{ms^2 + (b + K_d)\, s + K_p}$$

$$\Rightarrow ms^2 + (b + K_d)\, s + K_p = 0$$

$\boxed{K_p}$

$$\left[ms^2 + (b + K_d)\, s\right] + K_p = 0$$

$$1 + K_p \frac{1}{\underset{\substack{\uparrow \\ \text{not monic} \\ (\text{not } =1)}}{m} s^2 + (b + K_d)\, s} = 0$$

$$\Rightarrow 1 + \left(\frac{K_p}{m}\right) \frac{1}{s^2 + \left(\frac{b+K_d}{m}\right) s} = 0$$

$$K = \frac{K_p}{m}, \quad b(s) = 1, \quad a(s) = s^2 + \left(\frac{b + K_d}{m}\right) s$$

$\boxed{K_d}$

$$\left[ms^2 + bs + K_p\right] + K_d s = 0$$

$$1 + K_d \frac{s}{ms^2 + bs + K_p} = 0$$

$$\Rightarrow 1 + \left(\frac{K_d}{m}\right) \frac{s}{s^2 + \left(\frac{b}{m}\right) s + \frac{K_p}{m}} = 0$$

$$K = \frac{K_d}{m}, \quad b(s) = s, \quad a(s) = s^2 + \frac{b}{m} s + \frac{K_p}{m}$$

We can study the effects of changing either gain (one at a time), but we have to use different values of $a(s)$ and $b(s)$ to do this.

## 2 Thinking about the Root Locus in Terms of Magnitude and Phase

Taking a deeper look at $1 + K\frac{b(S)}{a(s)} = 0$, we find that

$$K\frac{b(s)}{a(s)} = K\frac{\prod\limits_{m}(s - z_i)}{\prod\limits_{n}(s - p_i)} \quad \text{is just a complex number.}$$

In order to satisfy the characteristic equation, we need to have

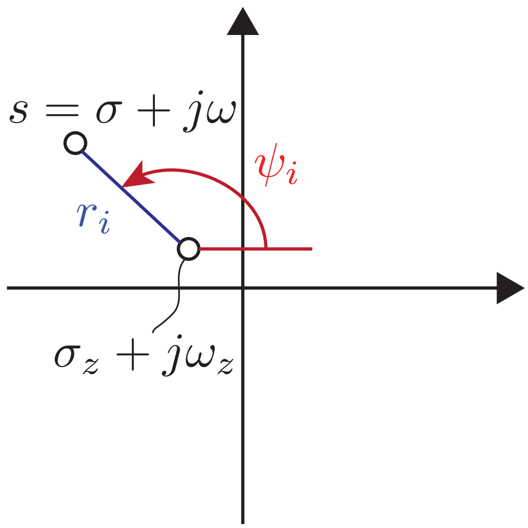
$$K\frac{b(s)}{a(s)} = -1$$

Therefore, this complex number has a magnitude of 1 and a phase of 180°.

So the root locus is the set of all points $s$ such that:

$$K\frac{b(s)}{a(s)} = -1, \quad \text{which means that} \quad \left|K\frac{b(s)}{a(s)}\right| = 1 \quad \& \quad \angle K\frac{b(s)}{a(s)} = 180°$$

We will now take a closer look at what points on the s-plane look like, and how we might relate them to one another. Specifically we will relate an arbitrary point $s$ to the location of a zero, $z_i$.




For any point $s = \sigma + j\omega$, $s - z_i$ is also a complex number.

$$\begin{aligned} s - z_i &= (\sigma - \sigma_z) + j\,(\omega - \omega_z) \\ &= r_i e^{j\psi_i} \end{aligned}$$

This is simply a way of expressing the distance and angle between a point $s$ and zero $z_i$.

Similarly, we can express the distance between the point $s$ and a pole $p_i$ as

$$s - p_i = l_i e^{j\phi_i}$$

With these complex distances expressed in exponential form, the root locus condition can be re-intrepreted.

$$1 = \left| K \frac{\prod\limits_m (s - z_i)}{\prod\limits_n (s - p_i)} \right| = \left| K \frac{\prod\limits_m \left( r_i e^{j\psi_i} \right)}{\prod\limits_n \left( l_i e^{j\phi_i} \right)} \right| = K \frac{\prod\limits_m r_i}{\prod\limits_n l_i}$$

This implies that for a point to lie on the root locus,

$$K \left( \frac{\text{product of distances to each zero}}{\text{product of distances to each pole}} \right) = 1$$

Note that this is not a *test* for a point to lie on the root locus. Rather, for a point that lies on the root locus, the above equation will tell you the value of $K$ for that point.

Another conclusion can be reached by looking at the angle:

$$\angle K \frac{\prod\limits_m r_i e^{j\psi_i}}{\prod\limits_n l_i e^{j\phi_i}} = \angle K \frac{\left( \prod\limits_m r_i \right) \left( e^{j\psi_1} e^{j\psi_2} \dots e^{j\psi_m} \right)}{\left( \prod\limits_n l_i \right) \left( e^{j\phi_1} e^{j\phi_2} \dots e^{j\phi_n} \right)}$$

$$= \angle \left( K \frac{\prod\limits_m r_i}{\prod\limits_n l_i} \right) \cdot \frac{e^{j(\psi_1 + \psi_2 + \dots + \psi_m)}}{e^{j(\phi_1 + \phi_2 + \dots + \phi_n)}}$$

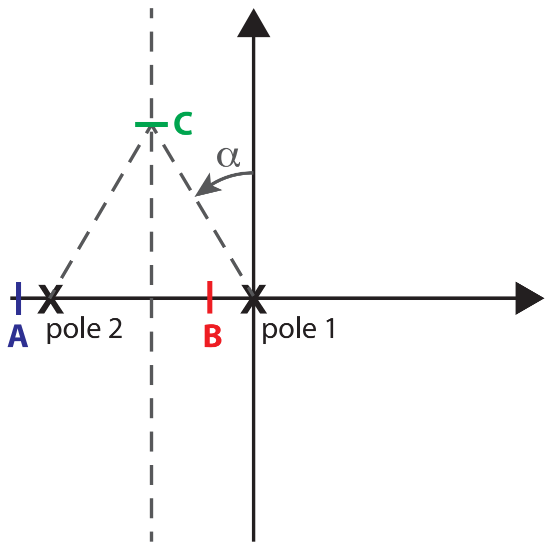
$$= \angle \frac{e^{j(\psi_1 + \psi_2 + \dots + \psi_m)}}{e^{j(\phi_1 + \phi_2 + \dots + \phi_n)}}$$

$$= \angle e^{j(\psi_1 + \psi_2 + \dots + \psi_m - \phi_1 - \phi_2 - \dots - \phi_n)}$$

$$= \underbrace{\sum_m \psi_i}_{\substack{\uparrow \\ \text{sum of} \\ \text{angles to} \\ \text{each zero}}} - \underbrace{\sum_n \phi_i}_{\substack{\uparrow \\ \text{sum of} \\ \text{angles to} \\ \text{each pole}}} = 180^\circ + 360^\circ (l - 1) \quad \text{for any integer } l = 1, 2, \dots$$

**Example**

Consider this system with two poles and no zeros. Check three test points A, B and C to see if they lie on the root locus:




A: $\phi_1 = 180°$ (line goes from pole to test point)

$\phi_2 = 180°$

$\sum \psi_i - \phi_i = -360° \neq 180°$

This point is *not* on the root locus

B: $\phi_1 = 180°$

$\phi_2 = 0°$

$\sum \psi_i - \phi_i = -180°$

on root locus

C: $\phi_1 = 90° + \alpha$

$\phi_2 = 90° - \alpha$

$\sum \psi_i - \phi_i = -180°$

on root locus

## 3 Root Locus Rules

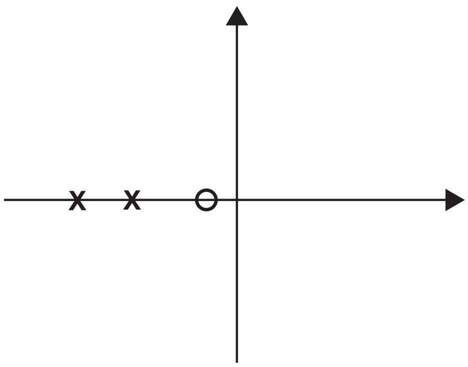
This means we can apply a simple check to any point in the plane to see if it lies on the root locus. However, we really don't want to check every point in the plane! We want some rules to use instead.

As we develop these rules, we often think of what happens as we increase the gain $K$ from 0 to infinity.

**Rule 1** $n$ branches of the locus start at poles $a(s) = 0$ (zero gain)
$m$ of these go to zeros $b(s) = 0$

Since $a(s) + Kb(s) = 0 \qquad K = 0 \Rightarrow a(s) = 0$

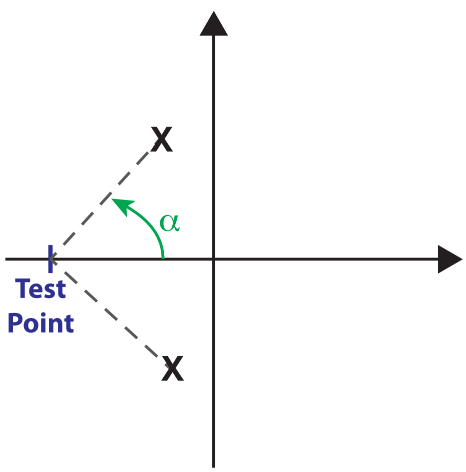
$$b(s) + \frac{1}{K}a(s) = 0 \qquad K \to \infty \Rightarrow b(s) = 0$$



If a system has 2 poles and 1 zero, it has two branches. One goes to the zero and one goes to infinity.

Which is which?

**Rule 2** Points on the real axis to the left of an odd number of poles and zeros are on the locus




On the real axis:

$\sum \psi_i - \sum \phi_i = 0$ for any complex conjugate pairs
$\Rightarrow$ These don't contribute

$\psi_i$ or $\phi_i = 180°$for any pole or zero to the right
Need an odd number of $180°$ contributions

So in the previous example: