ENGR 105: Feedback Control Design
Winter 2013

Lecture 16 - Root Locus Rules

Wednesday, February 13, 2013

Today's Objective

Define and derive rules for drawing the root locus

Reading: FPE Sections 5.1, 5.2

Introductory section if necessary

Given a characteristic equation in the form

$$1 + K\frac{b(s)}{a(s)} = 0$$

the root locus describes how the closed-loop poles change as K varies from zero to infinity.

A few basic rules make drawing a root locus rather straightforward.

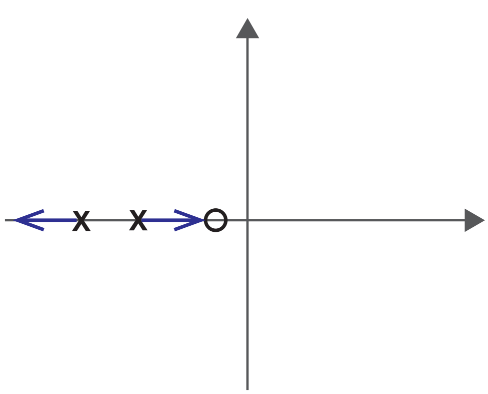
Rule 1

n branches of the root locus start at poles $a(s) = 0$ m of these to go to zeros $b(s) = 0$.

Rule 2

Points on the real axis to the left of an odd number of zeros and poles lie on the locus.

Example



2 poles, 1 zero
  1 pole goes to the zero
  1 poles goes to infinity

We have a rule describing *how* the poles go to infinity...

Rule 3

The $n - m$ poles go to infinity asymptotically along lines with angle

$$\phi_l = \frac{180^\circ + 360^\circ\,(l-1)}{n-m} \qquad l = 1, 2, \ldots, n-m$$

radiating from a point α

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

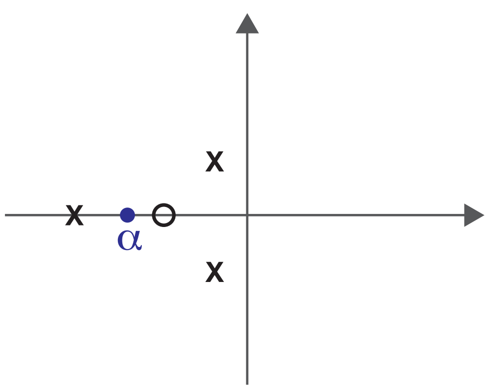
$$\text{Since } \frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n} = -\frac{1}{K},$$

as $K \to \infty$ either the poles $\to$ zeros or poles $\to \infty$

(If the closed-loop poles go to the zeros, the numerator is zero; if they go to infinity, so does the denominator).

Derivation

How can we derive this rule? Look at test points



- Near poles and zeros, the exact location of the test point relative to them is important
- But when the test point is very far away, the poles and zeros are seen as being clustered at one point. This means that our system looks like:

$$1 + K\frac{1}{(s-\alpha)^{n-m}} = 0$$

As test points are taken farther and farther away, it is only the net number of poles that becomes important. All of these can be treated as being at the single point α.

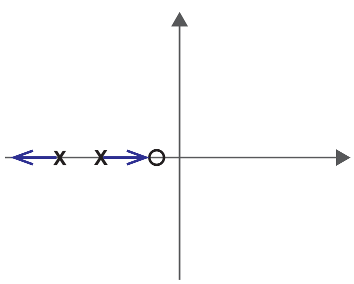
For this far-away test point lie to on the root locus, the total angle (sum of the $n-m$ angles which all have the same value, ϕ_l) from the poles to the test point is 180°

$$(n-m)\,\phi_l = 180^\circ + 360^\circ\,(l-1)$$

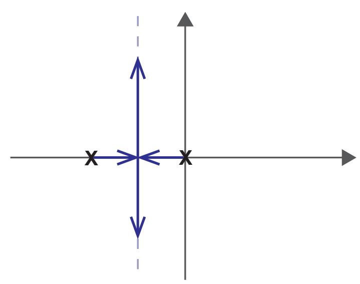
So to find the asymptotes, look at

$$\phi_l = \frac{180^\circ + 360^\circ\,(l-1)}{n-m} \qquad \text{for } l = 1, 2, \ldots n-m$$

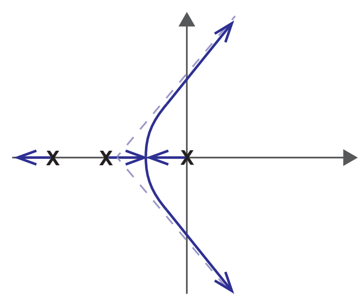
If $n-m=$ 1, single asymptote at 180°
2, two asymptotes at 90° and 270°
3, three asymptotes at 60°, 180°, and 300°



$n-m=1$



$n-m=2$



$n-m=3$

(this system will go unstable for high K!)

Centers of asymptotes

How do we find the centers of asymptotes (i.e., where asymptotes hit the real axis)?

The denominator polynomial is:

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = (s-p_1)\,(s-p_2)\ldots(s-p_n)$$

$$= s^n + a_1(-p_1 - p_2 - \cdots - p_n) + \ldots$$

$$\Rightarrow -a_1 = \sum_n p_i$$

Similarly, for the numerator:

$$s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots + b_m = (s - z_1)(s - z_2)\ldots(s - z_m)$$

$$\Rightarrow -b_1 = \sum_m z_i$$

If $n - m > 1$ (at least two extra poles)

$$\underset{\substack{\uparrow \\ \text{sum of roots is} \\ \text{independent of } K}}{s^n + a_1 s^{n-1}} + \ldots + a_n + K\left(s^m + b_1 s^{m-1} + \ldots + b_m\right) = 0$$

$$= (s - r_1)(s - r_2)\ldots(s - r_n)$$

$$\Rightarrow -a_1 = \underbrace{\sum p_i}_{\substack{\text{open loop} \\ \text{poles}}} = \underbrace{\sum r_i}_{\substack{\text{closed loop} \\ \text{poles}}}$$

$\Rightarrow$ sum of open loop poles = sum of closed loop poles

Interestingly, this is independent of the value of K

When K is large, roots either go to zeros or to the asymptotic system

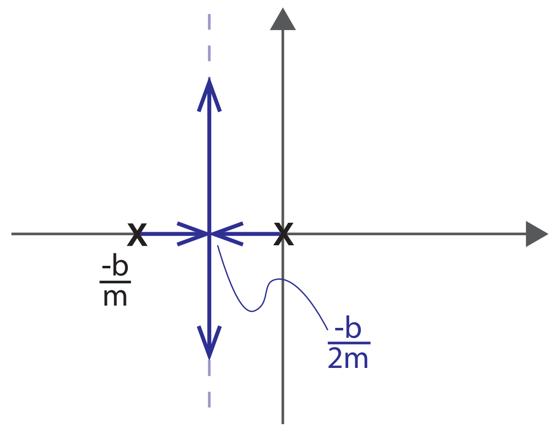
$$1 + K\frac{1}{(s - \alpha)^{n-m}} = 0$$

$$\sum r_i = \underbrace{(n - m)}_{\substack{n-m \text{ poles} \\ \text{at } \alpha}}\alpha + \underbrace{\sum z_i}_{\substack{m \text{ poles at} \\ \text{zero locations}}} = \sum p_i$$

So when $\alpha = \dfrac{\sum p_i - \sum z_i}{n - m}$, our asymptotic system has the correct behavior.

Examples

Returning to our example from earlier:



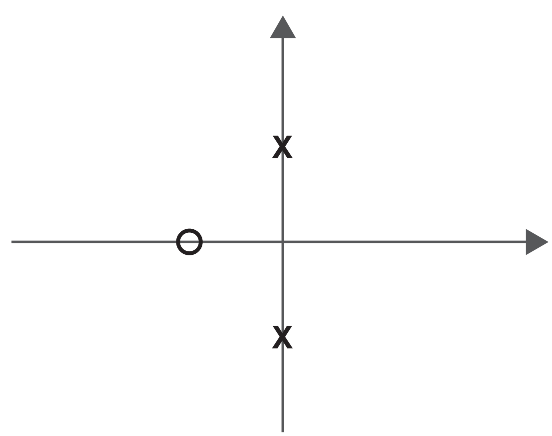
$$1 + K\frac{1}{s^2 + \frac{b}{m}s}$$

$$s = -\frac{b}{m}, 0$$

$$\alpha = \frac{-\frac{b}{m} + 0}{2} = -\frac{b}{2m}$$

Here we see that we have been able to draw the locus without needing to solve the quadratic equation or checking a lot of points.

What about the following system?



What does this system do? We know the real axis to the left of the zero lies on the root locus, but how do the poles get to the real axis?

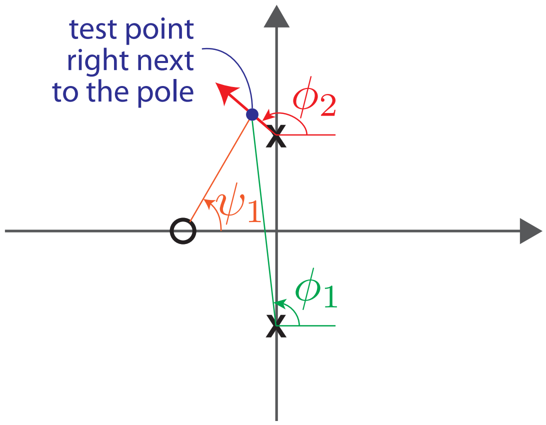
We need another couple of rules...

Rule 4

The angle of departure for a given pole is determined by examining a nearby test point that satisfies

$$\sum \psi_i - \sum \phi_i = 180°$$

where ψ_i is the angle from zero i to the test point and ϕ_i is the angle from pole i to the test point (measured with respect to the positive real axis).



At the test point

$$\angle\psi_1 - \angle\phi_1 - \angle\phi_2 = 180^\circ$$

$$\psi_1 - 90^\circ - \angle\phi_2 = 180^\circ$$

$$\Rightarrow \angle\phi_2 = -270^\circ + \psi_1 = 90 + \psi_1$$

So the pole departs at 90° from the line from the zero.

Where do the roots meet up? By symmetry, they hit the real axis at the same point. See FPE6 book (Section 5.2.1, page 130) for the arrival angles.

Rules 5 and 6

Multiple roots on the locus occur where

$$b\frac{da}{ds} - a\frac{db}{ds} = 0$$

and branches approach a point of q roots at an angle of

$$\frac{180^\circ + 360^\circ\,(l-1)}{q}$$

Why is this true?

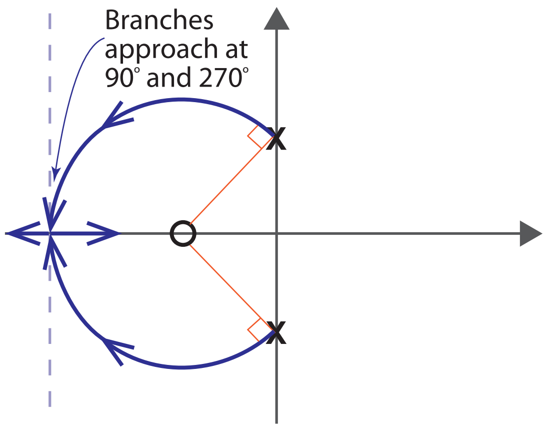
$$a(s) + K_1 b(s) = \underbrace{(s-r_1)^q}_{q \text{ roots at } s=r_1} \; f(s) = 0$$

$$\frac{da}{ds} + K_1\frac{db}{ds} = q\,(s-r_1)\,f(s) + (s-r_1)\,\frac{df}{ds}\bigg|_{s=r_1} = 0$$

$$K_1 = \frac{-a(s)}{b(s)}\bigg|_{s=r_1}$$

$$\Rightarrow b\frac{da}{ds} - a\frac{db}{ds}\bigg|_{s=r_1} = 0$$

Example



$$D(s)G(s) = \frac{s-z}{s^2+\omega^2} \qquad (z<0)$$

$$b = s - z \qquad \frac{db}{ds} = 1$$

$$a = s^2+\omega^2 \qquad \frac{da}{ds} = 2s$$

$$(s-z)\cdot 2s - \left(s^2+\omega^2\right)\cdot 1 = 0$$
$$2s^2 - 2sz - s^2 - \omega^2 = 0$$
$$s^2 - 2sz - \omega^2 = 0$$

$$s = \frac{2z \pm \sqrt{4z^2+4\omega^2}}{2}$$
$$= z \pm \sqrt{z^2+\omega^2}$$

Only $z - \sqrt{z^2+\omega^2}$ lies on the locus. This is a necessary condition, but the solutions need to be checked to see which one lies on the locus.

Please see the FPE6 textbook for thorough treatment of arrival angles.