ENGR 105: Feedback Control Design
Winter 2013

Lecture 18 - Frequency Response Basics

Monday, February 25, 2013

Today's Objectives

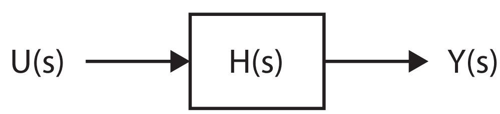
1. review the derivation of the the frequency response of a transfer function
2. revisit RC circuit example from a frequency response perspective
3. example comparing PD control and lead compensation

Reading: FPE Section 6.1

1 Review of Frequency Response

Thinking back to the concept of Fourier Transforms, a physical signal can be decomposed into a series of sinusoids at different frequencies. If we can describe what happens to each one of these sinusoids as it goes through a linear system, we can fully describe what happens to the signal. Thus, we can think of describing a linear system by its frequency response.

When we discussed transfer functions, we derived the sinusoidal response of a linear system:



$$U(s) = \frac{\omega A}{s^2 + \omega^2}$$

$$Y(s) = \underbrace{\frac{a_1}{s - p_1} + \frac{a_2}{s - p_2} + \ldots + \frac{a_n}{s - p_n}}_{\text{poles of } H(s)} + \underbrace{\frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega}}_{\text{sinusoidal response}}$$

Once the response from the stable poles dies out,

$$y(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}$$

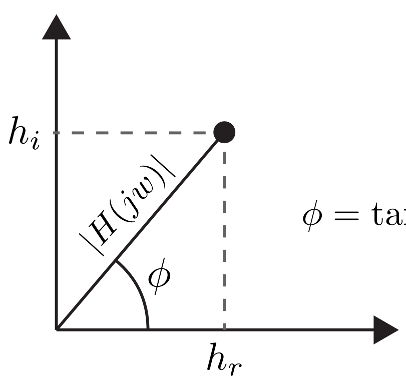
This can be rearranged into a more insightful form:

$$\begin{aligned} a &= (s + j\omega)\, Y(s)\Big|_{s=-j\omega} \\ &= (s + j\omega)\, H(s) \frac{A\omega}{s^2 + \omega^2}\Bigg|_{s=-j\omega} \\ &= H(s) \frac{\omega A}{s - j\omega}\Bigg|_{s=-j\omega} \\ &= -\frac{1}{2j} H(-j\omega) A \end{aligned}$$

Also,

$$\bar{a} = \quad \frac{1}{2j} H(j\omega) A$$

Remember, $H(j\omega)$ is just a complex number. It has real and imaginary components, which can also be described by a magnitude and phase:



$$\phi = \tan^{-1}\left(\frac{h_i}{h_r}\right)$$

$$\begin{aligned} H(j\omega) &= |H(j\omega)|\,(\cos\phi + j\sin\phi) \\ &= |H(j\omega)|\, e^{j\phi} \\ H(-j\omega) = \bar{H}(j\omega) &= |H(j\omega)|\, e^{-j\phi} \end{aligned}$$

$$y(t) = -\frac{1}{2j}\, |H(j\omega)|\, Ae^{-j\phi}e^{-j\omega t} + \frac{1}{2j}\, |H(j\omega)|\, Ae^{j\phi}e^{j\omega t}$$

$$= A\,|H(j\omega)|\, \frac{1}{2j}\left[e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right]$$

$$= A\,|H(j\omega)|\sin(\omega t + \phi) \rightarrow \text{steady-state relationship}$$

So a sine wave passed into a linear system produces a sine wave of the same frequency but different magnitude and phase (once transients have died out). The transfer function describes the change in the magnitude and phase.

$$u = A\sin\omega t \rightarrow \boxed{H(j\omega)} \rightarrow y = A\left|H(j\omega)\right|\sin\left(\omega t + \phi\right)$$

If I have a more complicated input that I can write as a sum of sinusoids, the output is a simple sum of sinusoids:

$$u = \sum_{i=1}^{n} A_i \sin\omega_i t \rightarrow \boxed{H(j\omega)} \rightarrow y = \sum_{i=1}^{n} A_i\left|H(j\omega_i)\right|\sin\left(\omega_i t + \phi_i\right)$$

$$\text{where } \phi_i = \tan^{-1}\left(\frac{\mathrm{Im}\left[H(j\omega_i)\right]}{\mathrm{Re}\left[H(j\omega_i)\right]}\right)$$

We can think of this in two ways:

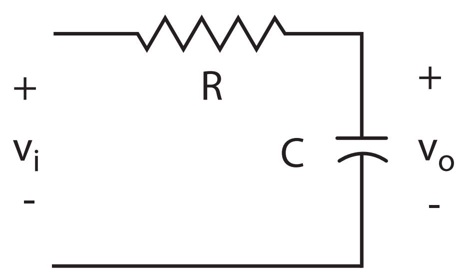
1. the transfer function fully describes the frequency response of the system
2. if we know the frequency response of the system, we can build the transfer function

In frequency-domain system identification, sinusoidal inputs are used to build an empirical transfer function experimentally. This can be an extremely useful method of obtaining a system model.

The frequency domain can be very useful in thinking about systems with vibrating or oscillations at specific frequencies. Audio systems and structural vibrations (in a car or bridge, for instance) are good examples.

We can use these techniques to gain insight into systems we have already studied.

2 Example: RC circuit revisited



$$\frac{V_0(s)}{V_i(s)} = \frac{1}{RCs+1} = H(s)$$

Magnitude: $|H(j\omega)| = \dfrac{1}{\sqrt{R^2C^2\omega^2+1}}$

DC gain ($\omega = 0$): $H(0) = 1$

$|H(j\omega)| \to 0$ as $\omega \to \infty$ (low pass filter)

$|H(j\omega_c)|$ for $\omega_c = \dfrac{1}{RC}$: $|H(j\omega_c)| = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

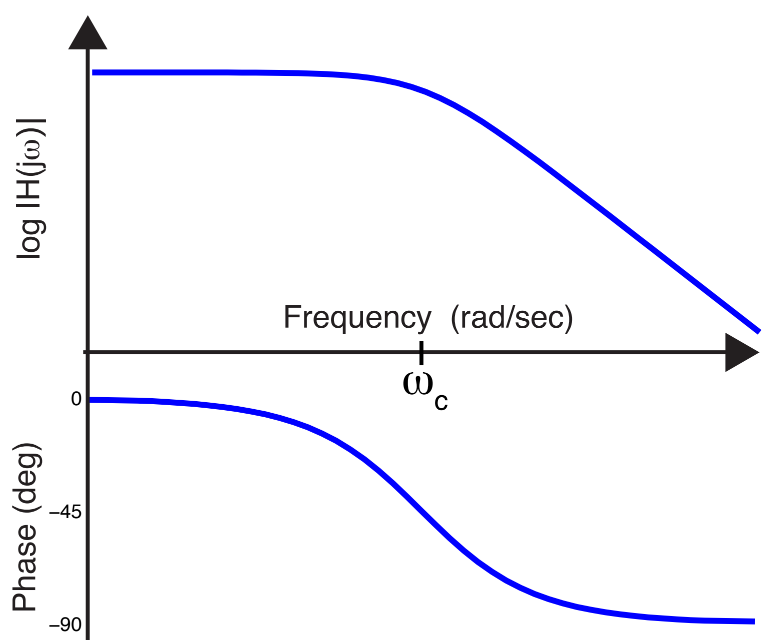
Phase: $H(j\omega) = \dfrac{1}{RCj\omega+1} = \dfrac{1-j\omega RC}{1+\omega^2R^2C^2} = \dfrac{1}{1+\omega^2R^2C^2} - j\dfrac{\omega RC}{1+\omega^2R^2C^2}$

$\angle H(0) = 0°$

As $\omega \to \infty, \angle H(j\omega) \to -90°$ Get this from $\tan^{-1}(-j\omega RC)$

$\angle H(j\omega_c) = \angle\left(\dfrac{1}{2} - \dfrac{j}{2}\right) = -45°$

So our system response looks like:



3 Example: PD control and lead compensation compared

Lead compensator: $D(s) = K\frac{Ts+1}{\alpha Ts+1}$ $\quad \substack{0<\alpha<1 \\ K>0}$

$$|D(j\omega)| = |K|\frac{|Tj\omega+1|}{|\alpha Tj\omega+1|} = |K|\frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\alpha\omega T)^2}}$$

$|D(j\omega)| \approx K$ at low frequencies

$$|D(j\omega)| = K\frac{\sqrt{\frac{1}{\omega^2}+T^2}}{\sqrt{\frac{1}{\omega^2}+\alpha^2T^2}} \approx \frac{K}{\alpha} \text{ at high frequencies}$$

$$\phi = \angle(1+j\omega T) - \angle(1+\alpha j\omega T)$$

$$= \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

$\phi = 0^\circ$ at low frequencies and high frequencies; in between, $\phi > 0^\circ$.

PD Control: $D(s) = K(Ts+1)$

$D(j\omega) = K(Tj\omega+1)$ same as lead compensator when $\alpha = 0$

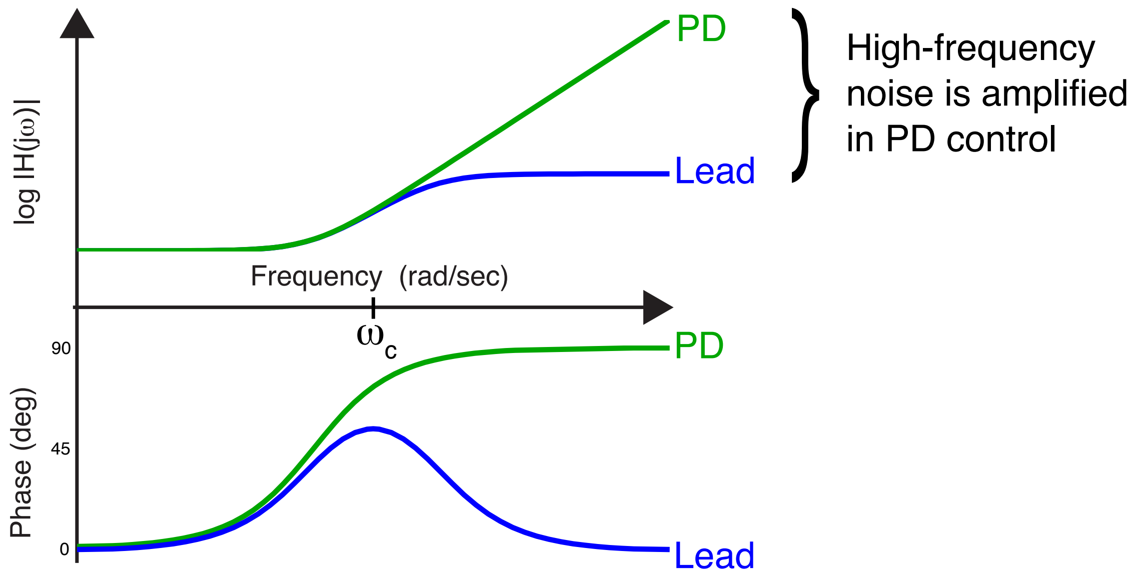
$|D(j\omega)| = K$ at low frequencies

$|D(j\omega)| \to \infty$ as $\omega \to \infty$

$\phi = \angle(1+j\omega T) = 0^\circ$ at $\omega = 0$

$\phi \to 90^\circ$ as $\omega \to \infty$

Graphically:



This gives a clearer explanation of lead compensation as a combination of PD control and a low-pass filter. This is literally true. The lead compensator does not amplify high-frequency signals as much as the PD controller. By changing the value of α, we can make the lead compensator match the response (and behavior) of the PD controller over a certain frequency range.

Our intuition is that signals beyond the frequency range of interest represent "noise" that we do not want to amplify.