ENGR 105: Feedback Control Design
Winter 2013

Lecture 20 - Bode and Stability

Friday, March 1, 2013

Today's Objectives

1. re-examine neutral stability from the point of view of root locus and Routh array
2. neutral stability from Bode plots
3. introduce the concepts of gain margin and phase margin

Reading: FPE Section 6.2, 6.4

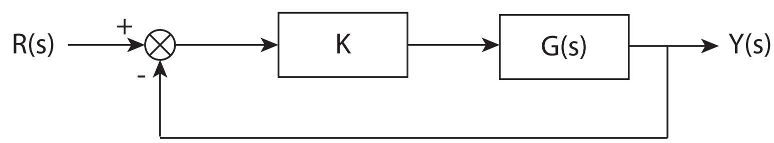
1 Neutral Stability (from root locus and Routh array)

Neutral stability (also known as marginal stability) tell us about the transition of a system from stable to unstable as a gain K of the system changes.

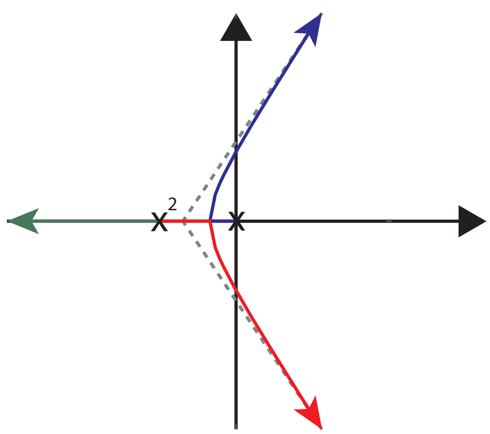
Consider the following plant with a pole at the origin:

$$G(s) = \frac{1}{s\,(s+1)^2}$$

Say we place this plant in a unity gain, negative feedback control loop with a simple proportional controller, K:



Root Locus:



small gain → stable

large gain → unstable

$|KG(s)| = 1$
$\angle G(s) = 180°$

at the point of neutral stability, $s = j\omega$:

$|KG(j\omega)| = 1$
$\angle G(j\omega) = 180°$

Routh Array:

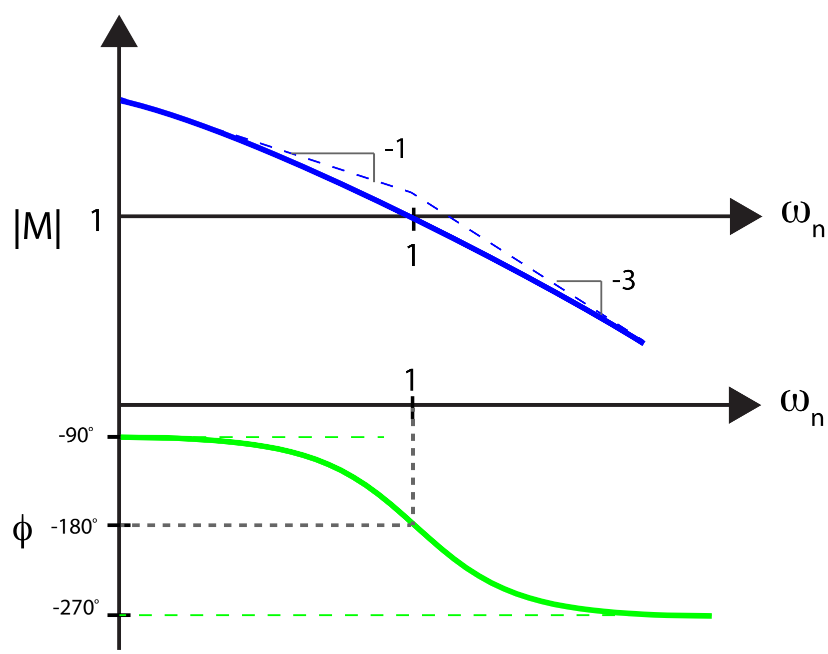
$$1 + \frac{K}{s(s+1)^2} = 0$$
$$s^3 + 2s^2 + s + K = 0$$
$$\begin{cases} 1 & 1 \\ 2 & K \\ \frac{K-2}{-2} & 0 \\ K & \end{cases} \rightarrow \quad \begin{matrix} K > 0 \\ K < 2 \end{matrix} \quad \text{for stability}$$

If $K > 2$, we end up with 2 sign changes, so we would have 2 poles go unstable. Two sign changes in the Routh array are characteristic of an oscillatory unstable response.

2 Neutral Stability (from Bode plots)

However, there is an alternative approach to using the Routh array (or more generally, finding the roots of the closed-loop characteristic equation): You can determine whether increasing of decreasing the gain K from the point of neutral stability increases or decreases the system's stability. This can be found from the Bode plots, and may be useful because (1) you might already have a Bode plot, and do not want to bother with the Routh array, and (2) in many practical problems, you measure the frequency response of the system directly; you might have an experimental Bode plot but not know the transfer function.

Bode plot at $K = 2$:



Notes on finding this Bode plot: $KG(s) = K_0 \frac{1}{s} \frac{1}{(s+1)} \frac{1}{(s+1)}$ where $K_0 = 2$, and in the terminology of the previous lecture, $\tau = 1$. For the pole at the origin, the magnitude asymptote slope is -1 (or -20 dB/decade) and the asymptote is shifted up by $K_0 = 2$. For each of the two poles at $s = -1$, the magnitude asymptote for low frequencies is horizontal, and for frequencies above the "cutoff" frequency ($\omega = 1$ rad/s), the magnitude asymptote has a slope of -1 (or -20 dB/decade). For the phase, the pole at the origin contributes -90° for all frequencies. Each pole at $s = -1$ contributes 0° at low frequencies, -90° at high frequencies, and -45° at $\omega = 1$. Adding all these contributions together gives the Bode plot sketched on the previous page.

We have the asymptotes, but what exactly is the magnitude at $\omega = 1$?

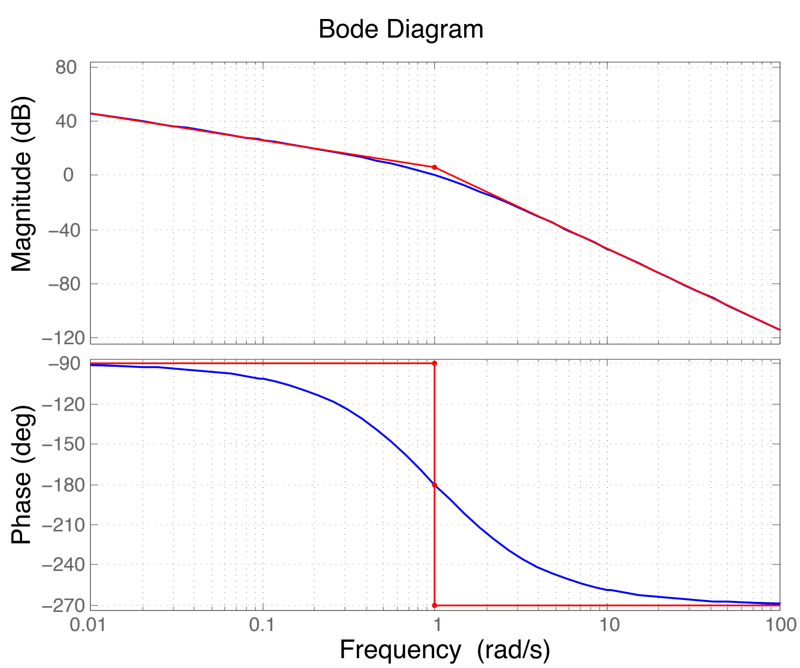
$\frac{2}{s} \rightarrow \left|\frac{2}{j\omega}\right| = 2$ and $\frac{1}{(s+1)^2} \rightarrow \left|\frac{1}{(j\omega+1)^2}\right| = \frac{1}{2}$

Thus, total magnitude of $KG(j\omega)$ at $\omega = 1$ is $2 \cdot \frac{1}{2} = 1$. The *Bode gain* (i.e., when $K = 1$) is $\frac{1}{2}$.

MATLAB Note: If you wanted to compute the Bode plots in MATLAB, here are a set of commands you could use:

```
sys = tf([2],[1 0])*tf([1],[1 1])*tf([1],[1 1])
bode(sys)
grid on
```

You can also check out user-contributed MATLAB functions that plot asymptotes. For example, `asymp(sys)` gives:



So what does this tell us about stability? When the points on the root locus become neutrally stable, they are purely oscillatory, so $s = j\omega$. We also know that since this "crossing point" is on the root locus, it must satisfy $|KG(s)| = 1$, and $\angle G(s) = 180°$.

We can look at the neutral stability point in two ways. First, if we know that $K = 2$ results in neutral stability, we can compute the ω at which this occurs:

$$1 + KG(s) = 0$$

$$(j\omega)^3 + 2(j\omega)^2 + (j\omega) + K = 0$$

$$-j\omega^3 - 2\omega^2 + j\omega + 2 = 0$$

$$\omega = 1 \text{ works, so this is the frequency at crossing}$$

Second, you could know that the crossing occurs at $\omega = 1$ (from the Bode phase plot) and then compute the K that makes this happen.

$$@\ \omega = 1, \text{ Bode gain } = \frac{1}{2} \text{ for } G(s)$$

$$|KG(s)| = 1 \rightarrow K\frac{1}{2} = 1$$

$$\therefore K = 2 \text{ for neutral stability}$$

Note that this analysis is appropriate *only* when a system is stable for low gains and unstable for high gains, with minimum phase. This limitation also applies to the gain margin and phase margin definitions we will explore next. (See page 319 of the textbook for a brief discussion of different cases.)

3 Gain margin and Phase margin

Gain margin (GM) is the factor by which the gain can be raised (from the value of K for which the Bode plot is drawn) before instability results. It can be read directly from the Bode plot by measuring the vertical distance between the $|KG(j\omega)|$ curve and $|KG(j\omega)| = 1$ line at the frequency where $\angle G(j\omega) = 180°$ (or $-180°$).

Phase margin (PM) is the amount by which the phase of $G(j\omega)$ exceeds -180° when $|KG(j\omega)| = 1$.

Why does this work? For gain margin, consider the requirement that neutral stability must occur at the frequency when $\angle G(j\omega) = 180°$ (this is independent of K). At that frequency, K·Bode gain = 1. So the gain margin essentially measures the Bode gain at that frequency for the existing value of K, and asks how much that K must be scaled in order to achieve (or lose) stability.

For phase margin, consider that when $\phi = -180°$, the system becomes unstable by satisfying the root locus criteria. When $|G(s)| = 1$, the amount of phase left before we hit $-180°$ is the phase margin. For a second-order closed-loop transfer function, there exists an approximate relationship between damping coefficient and PM:

$$\zeta \approx \frac{\text{PM}}{100} \qquad \text{(up to about } 60° \text{ of PM)}$$

PM $\to 0$ as damping $\to 0$.

Here is a Bode plot of $G(j\omega)$ (like our earlier Bode plot, but for $K = 1$) from Figure 6.35 of FPE6, with GM and PM labeled: