ENGR 105: Feedback Control Design
Winter 2013

Lecture 21 - Nyquist Stability

Monday, March 4, 2013

Today's Objectives

1. review the neutral stability criterion
2. explain Cauchys Argument Principle
3. show how this applies to stability
4. simple example of sketching the Nyquist plot and checking for stability

Reading: FPE Sections 6.2, 6.3

1 Neutral Stability

The concept of neutral stability discussed in the last lecture is useful in examining the range of stability for systems with no right half plane zeros or poles. It shows that there is a particular significance to points defined by:

$$|KG(j\omega)| = 1 \qquad \text{at} \qquad \angle KG(j\omega) = \angle G(j\omega) = 180^\circ$$

We know that these are conditions such that $s = j\omega$ lies on the root locus. In other words, this occurs when the system has a pole on the imaginary axis. However, this stability criterion only holds for a subset of systems: those for which increasing gain leads to instability, and $|KG(j\omega)|$ crosses the magnitude 1 once.

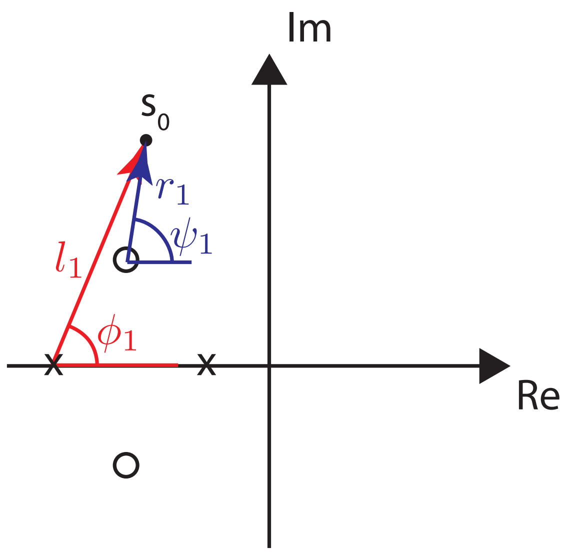
For more complex systems, it isn't immediately clear whether the point of neutral stability represents the system transitioning from stable to unstable as the gain is increased or the other way around. It also is not clear how to handle multiple crossings of the imaginary axis.

Harry Nyquist of Bell Laboratories (yes, another Bell Labs person!) worked this out in 1932 using Cauchy's Argument Principle.

2 Argument Principle

Remember that if we want to evaluate a transfer function $H(s)$ anywhere in the complex plane, we can look at the angles and distances to the poles and zeros.

Consider a test point s_0.



We can evaluate our transfer function $H(s)$ at this test point in terms of its magnitude and phase:

$$|H(s_0)| = \frac{r_1 \cdot r_2 \cdot \ldots}{l_1 \cdot l_2 \cdot \ldots}$$

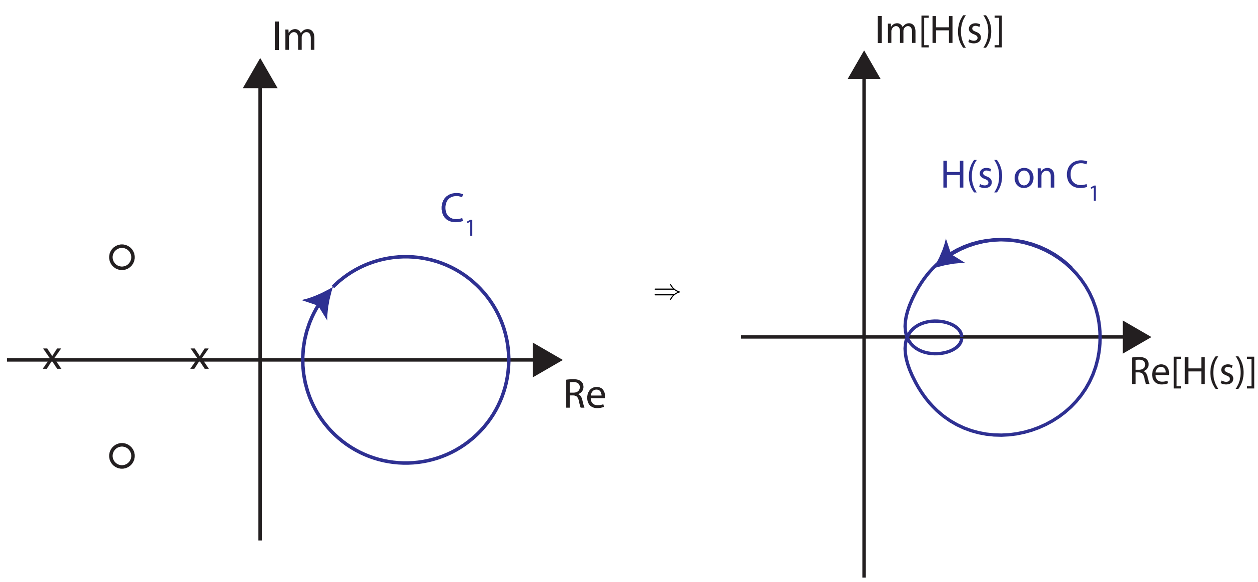
$$\angle H(s_0) = \sum \psi_i - \sum \phi_i$$

We can write

$$H(s_0) = |H(s_0)| e^{j\angle H(s_0)}$$

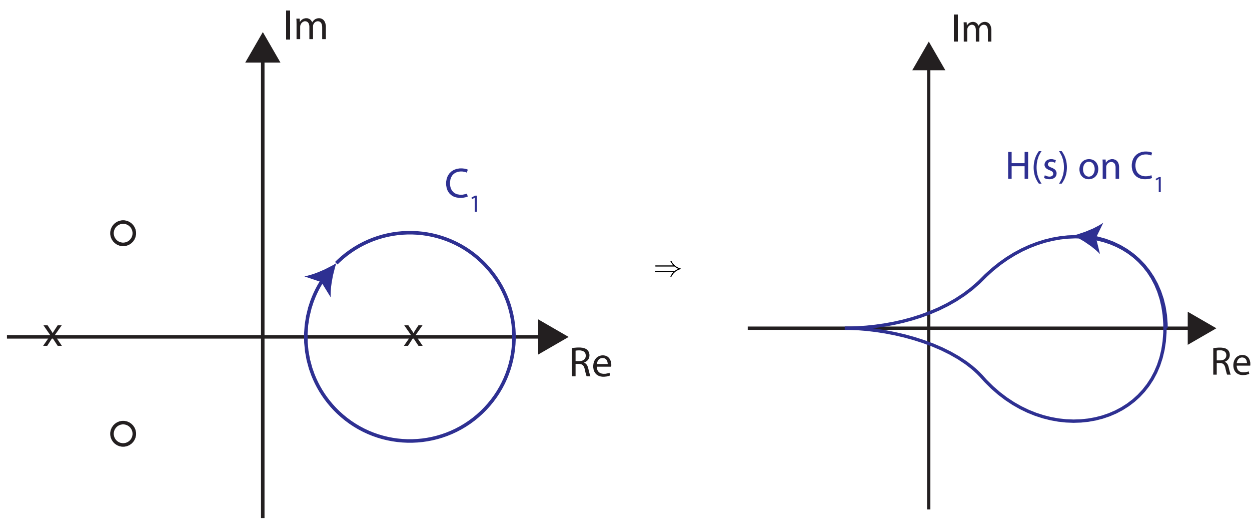
.

Now suppose we want to evaluate $H(s)$ around a clockwise contour (or loop) C_1 in the s-plane.



This can be done point by point for each test point on the contour C_1. The plot on the right is known in controls as a Nyquist plot. You can think of it as the real and imaginary components of $H(s)$ (corresponding the vertical and horizontal axes), or as a *polar plot* of angles $\angle H(s)$ and magnitudes $|H(s)|$, evaluated as s moves around a clockwise contour.

A interesting observation in this example is that $H(s)$ evaluated on C_1 never encircles the origin. This is because none of the poles or zeros of $H(s)$ are contained in C_1. If they were, we would get a different picture:



This is because the points on the contour now make a complete 360° rotation with respect to any point outside the contour. Consider

$$\angle H(s_0) = \psi_1 + \psi_2 - \phi_1 - \phi_2.$$

As we travel around the contour C_1, the angles ψ_1, ψ_2, and ϕ_1 (corresponding to the two zeros and the pole *outside* the contour) will not undergo a net change. But the angle between the test points around the contour and the pole *inside* the contour will undergo a net change of -360° after one full traverse of C_1. This causes $H(s)$ evaluated on C_1 to encircle the origin in the counterclockwise direction.

Cauchy's argument principle states that:

> The polar plot of $H(s)$ will circle the origin $Z - P$ times,
>
> where Z is the number of zeros inside the contour
>
> and P is the number of poles inside the contour.

These are clockwise encirclements, so a counterclockwise encirclement is the same as -1 clockwise encirclements. The negative sign on P comes from the $\angle H(s_0)$ equation above, which in turn came from the fact that the poles were in the denominator of the transfer function.

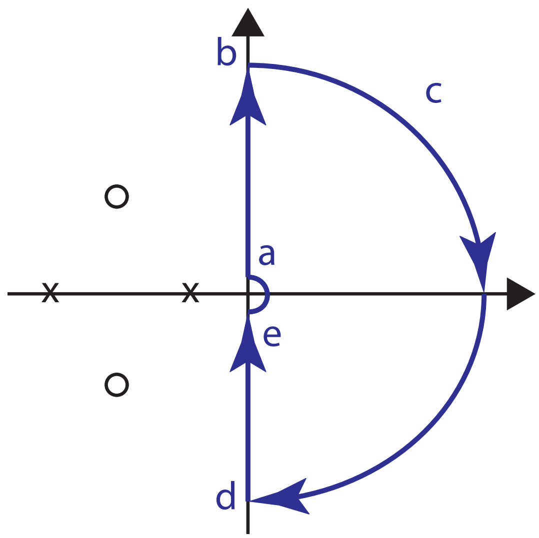
3 Stability

How can we use the Argument Principle to say something about system stability?

For stability, we want to evaluate if we have any closed-loop poles in the right half plane (for some value of K). Therefore we want to evaluate our closed-loop characteristic equation

$$1 + KG(s)$$

using a clockwise contour corresponding to the *entire* right half plane. (If we have poles or zeros that lie on the imaginary axis, we will take a small detour around them, as will be shown later in an example.)



Before formulating the formal stability criterion, we make two more observations:

(1) The zeros of $1 + KG(s)$ are the closed-loop poles
The poles of $1 + KG(s)$ are the open-loop poles

This follows from: $1 + KG(s) = 0$

$$1 + K\frac{b(s)}{a(s)} = 0$$

$$\frac{a(s) + Kb(s)}{a(s)} = 0$$

$$a(s) + Kb(s) = 0$$

(2) $1 + KG(s)$ is just $KG(s)$ shifted to the right (along the real axis) by 1.

Thus, plotting $1 + KG(s)$ and looking at encirclements of the origin is the same as plotting $KG(s)$ and looking at encirclements of -1.

Putting this together gives the *Nyquist Stability Criterion*:

(1) Evaluate $KG(s)$ on the contour enclosing the right half plane

(2) Count the number of clockwise encirclements of -1 (call this N)

(3) Count the number of unstable open loop poles (call this P)

(4) The number of unstable closed-loop poles is $N + P$

In other words, if the plot of $KG(s)$ encircles -1, then $1+KG(s)$ must encircle 0 (the origin). This means that there must be zeros or poles of $1 + KG(s)$ (open- or closed-loop poles of our system) in the right half plane.

This may seem a bit abstract, but it pretty simple if you are systematic and work from the Bode plot of the system.

4 A Simple Example

Consider the closed-loop characteristic equation $1 + KG(s) = 0$ where

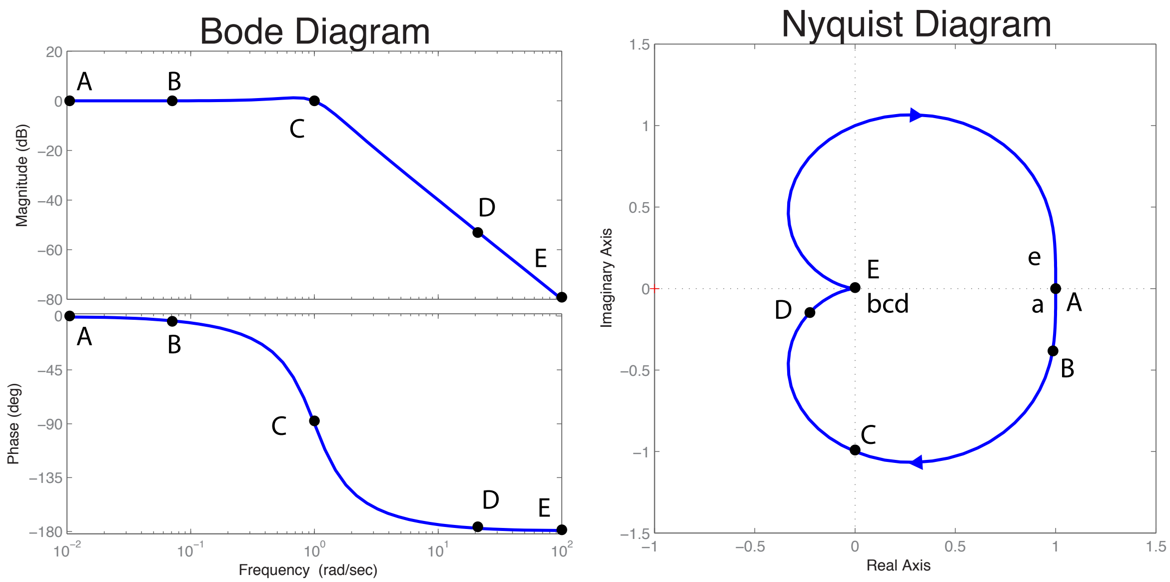
$$KG(s) = \frac{K{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Let's go through the steps to determine Nyquist stability for this system.

(1) Evaluate $KG(s)$ on the contour enclosing the right half plane:

The Bode plots give us this information! The Bode plots tell us the magnitude and phase of $KG(s)$ as $s = j\omega$ goes from $s = 0$ to $s = j\infty$. Because $G(j\omega)$ is the complex conjugate of $G(-j\omega)$, we can get the $s = -j\infty \to 0$ portion by reflecting about the real axis.

This takes us from the top of the imaginary axis to the bottom of the imaginary axis ($s = j\omega$ going from $+j\infty$ to $-j\infty$). But what about the encircling of the right half plane? Any $KG(s)$ that represents a physical system will have zero response at infinite frequency (i.e., has more poles than zeros). Thus, the big arc of the contour about the right half plane (which corresponds to s at infinity, now with both real and imaginary parts) results in $KG(s)$ being a point of infinitely small value near the origin for that part of the contour. So practically speaking, we can get all the information we need from just the Bode plots.



MATLAB code used to generate these plots:

```
K = 1;
zeta = .5;
wn = 1;
sys = tf([K*wn^2],[1 2*zeta*wn wn^2]);
figure('units','normalized','position',[0,0,.75,.5])
subplot(1,2,1)
bode(sys)
subplot(1,2,2)
nyquist(sys)
```

(2) Count the number of clockwise encirclements of -1 (call this N):

There are no encirclements of -1, so $N = 0$.

(3) Count the number of unstable open loop poles (call this P):

There are no unstable open-loop poles, so $P = 0$.

(4) The number of unstable closed-loop poles is $N + P$:

$N + P = 0$, so this system is stable.

This is true for the value $K = 1$ we used in our MATLAB code. But would it be true for any K?