ENGR 105: Feedback Control Design

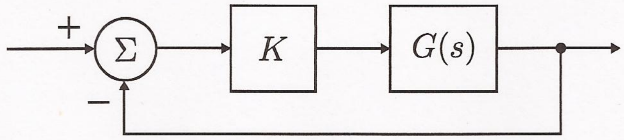
Final Exam

Winter Quarter 2011

Problem 1 (20 points)

$$G(s) = \frac{(s-10)}{(s+10)(s^2-2s+26)}$$

Assume that we are closing a feedback loop around this system in our usual unity-feedback form with a positive gain K (the usual 180° root locus conditions):



We want to find the root locus for this system.

(a) Find the locations of the poles and zeros. Plot this on these on the axes on the next page.

Zero: $s = 10$

poles: $s = -10,\ s = 1 \pm 5j$

$s^2 - 2s + 26 = 0$

$s = \frac{2 \pm \sqrt{4-104}}{2} = \frac{2 \pm 10j}{2}$

$= 1 \pm 5j$

(b) Sketch the portion of the real axis that belongs on the root locus. Calculate the center of asymptotes and estimate the angle of departure for the complex poles.

$\alpha = \frac{\sum P_i - \sum z_i}{n-m}$, $n=3,\ m=1$

$= \frac{(-10)+(1+5j)+(1-5j)-10}{2}$

$= \frac{-10-10+2}{2} = -9$

For $1+5j$:

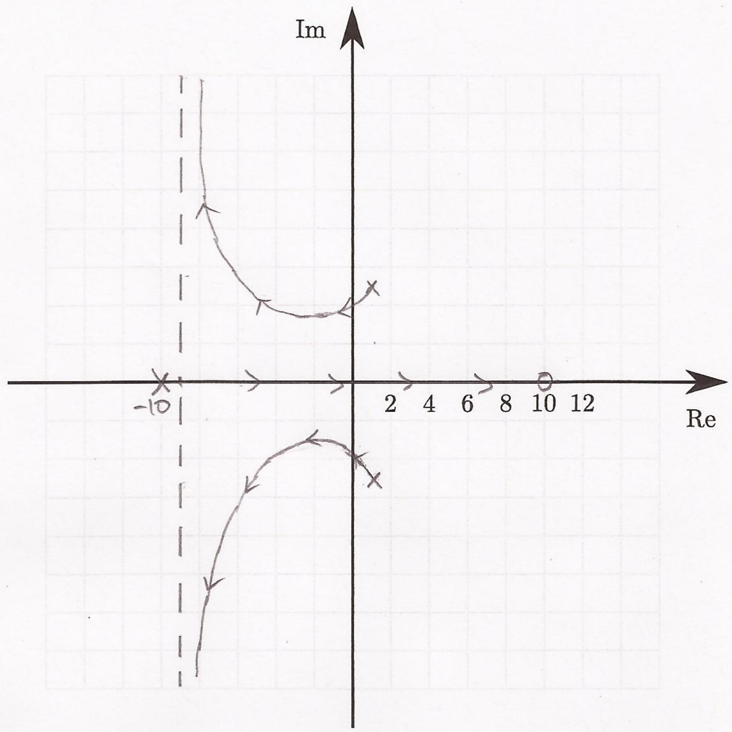
$\phi_{dep} = \sum \psi_i - \sum \phi_i - 180° - 360°(l-1)$

$= (90°+61°) - (90°+21°) - 180° - 360°(l-1)$

$= 61° - 21° - 180°$ $\quad l = 1$

$= -140°$ ($\phi_{dep} = -220°$ for $1+5j$ by symmetry)

(c) You should be able to come up with two plausible versions of the root locus depending upon which of the poles go to the asymptotes. Sketch both possibilities. The number of poles that cross the real axis may be a bit unclear at this point, so we will take a different look at the system.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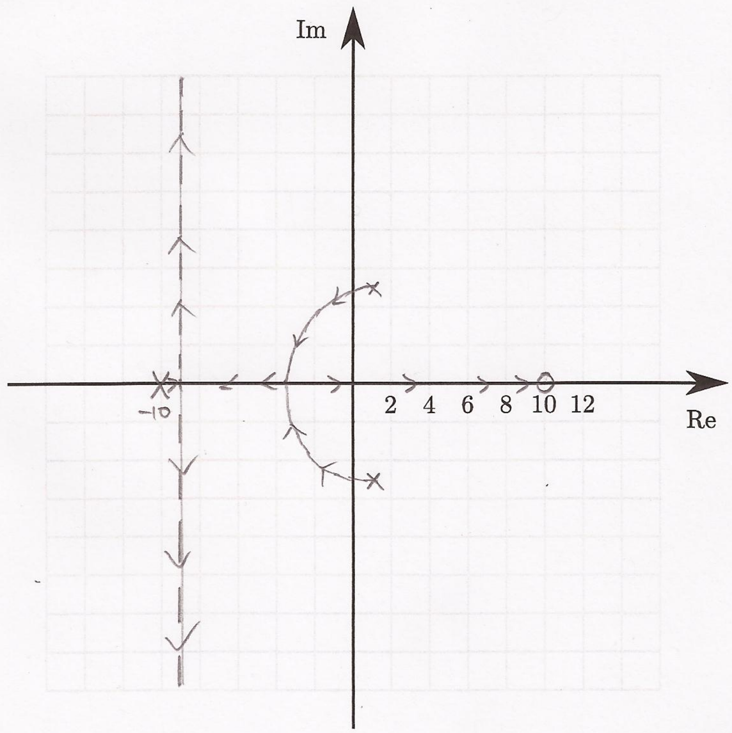
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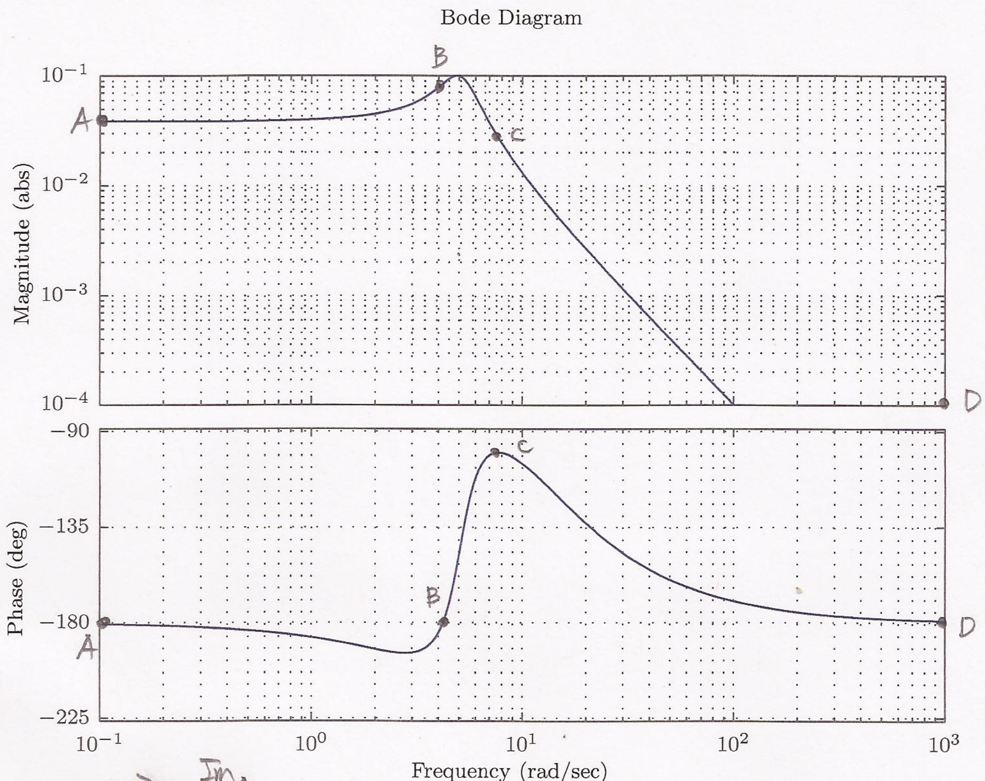
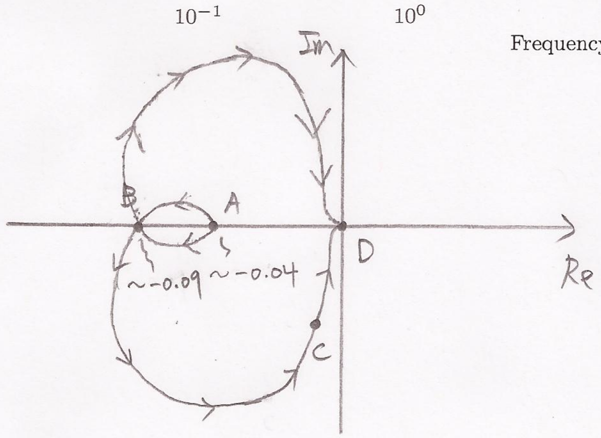
(d) Here is the Bode plot of the system. Sketch the Nyquist plot, using the Bode plot to label all important points quantitatively. (The magnitude plot continues at the same slope for the last decade of frequency – the axes are chosen to make the behavior at lower frequencies clear)





(e) Describe the location of the closed-loop poles as the gain varies from 0 to infinity. Since the you have the Bode plot, give values of the gain at which any transition in stability takes place.

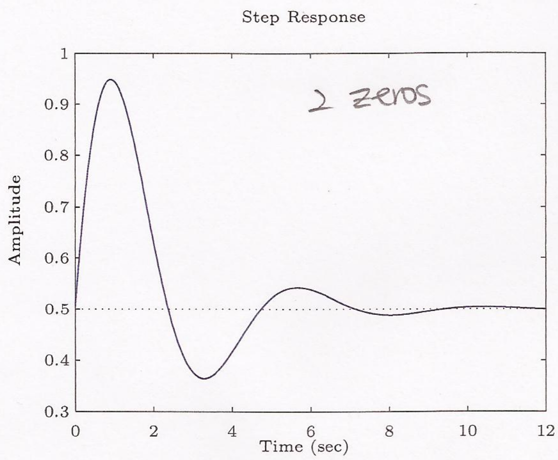
$0 < K < 11$, $N = 0$, $P = 2 \Rightarrow Z = N + P = 2 \Rightarrow$ 2 RHP poles, unstable

$11 < K < 25$, $N = -2$, $P = 2 \Rightarrow Z = N + P = 0 \Rightarrow$ 0 RHP poles, stable

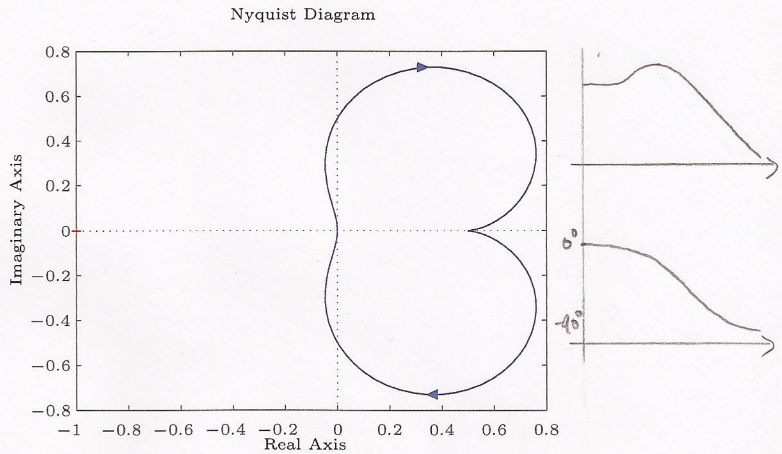
$25 < K < \infty$, $N = -1$, $P = 2 \Rightarrow Z = N + P = 1 \Rightarrow$ 1 RHP pole, unstable

Problem 2 (25 points)

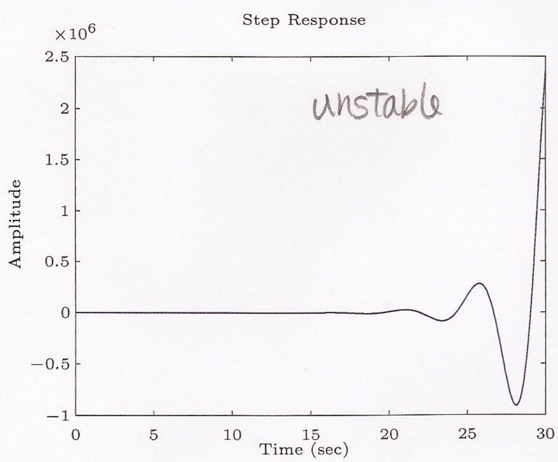
The following plots show open-loop unit step responses and a Nyquist plot for each of four **second-order systems**. Match the step response with the Nyquist and explain the features of the system that allows you to identify one with the other. (If you get stuck, remember the relationship between the Nyquist and the Bode plot – none of these plots have loops at infinity to make your life difficult)



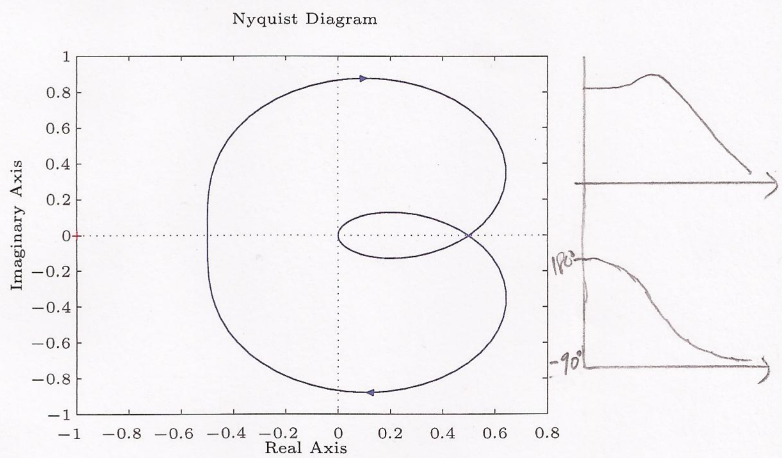
I



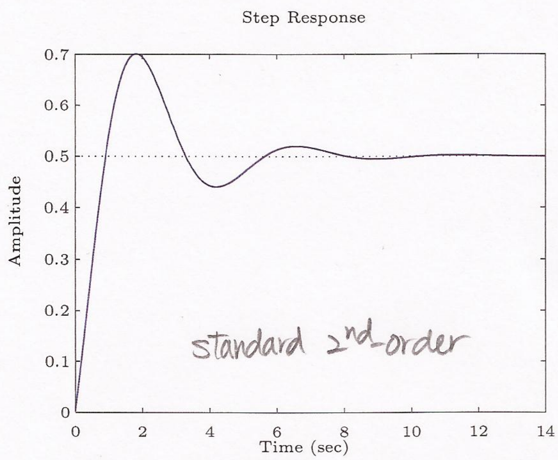
a



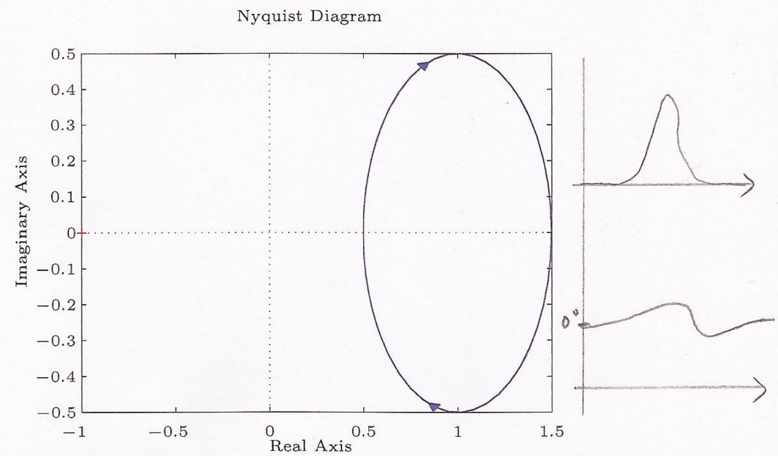
II



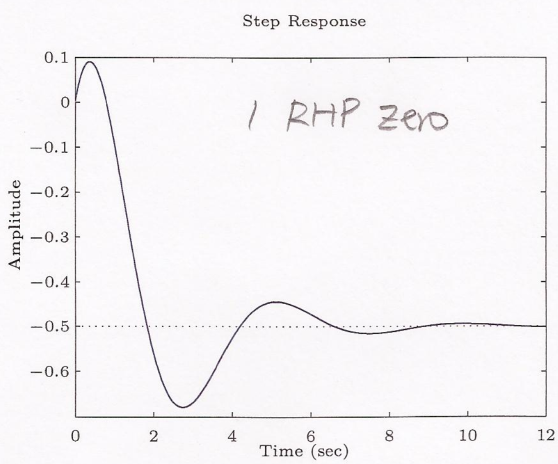
b



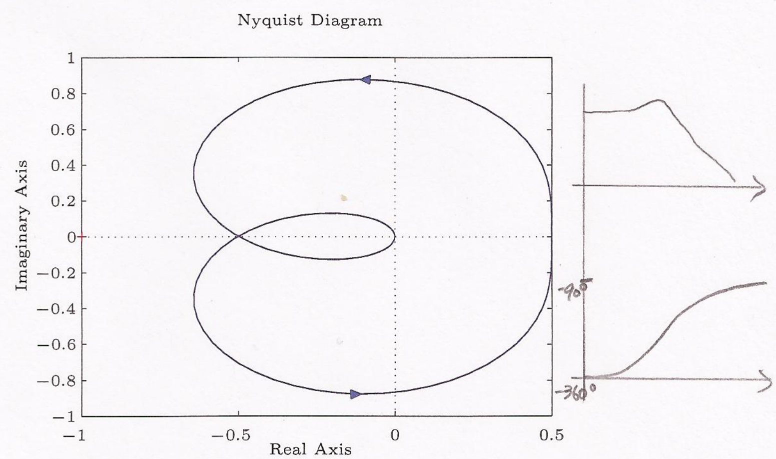
III



c



IV



d

Step response I: c

The phase starts and ends at 0°, suggesting a relative degree of 0. The step response starts at a non-zero value for the same reason.

Step response II: d

The phase keeps increasing even though there are poles.

Step response III: a

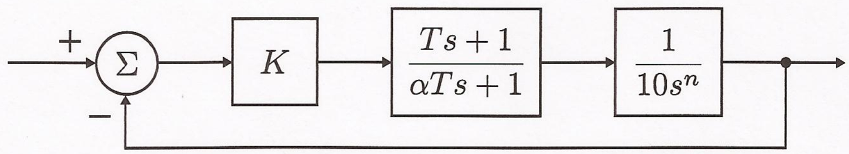
The phase plot starts at 0° and ends at −90°, suggest a LHP zero and two LHP poles.

Step response IV: b

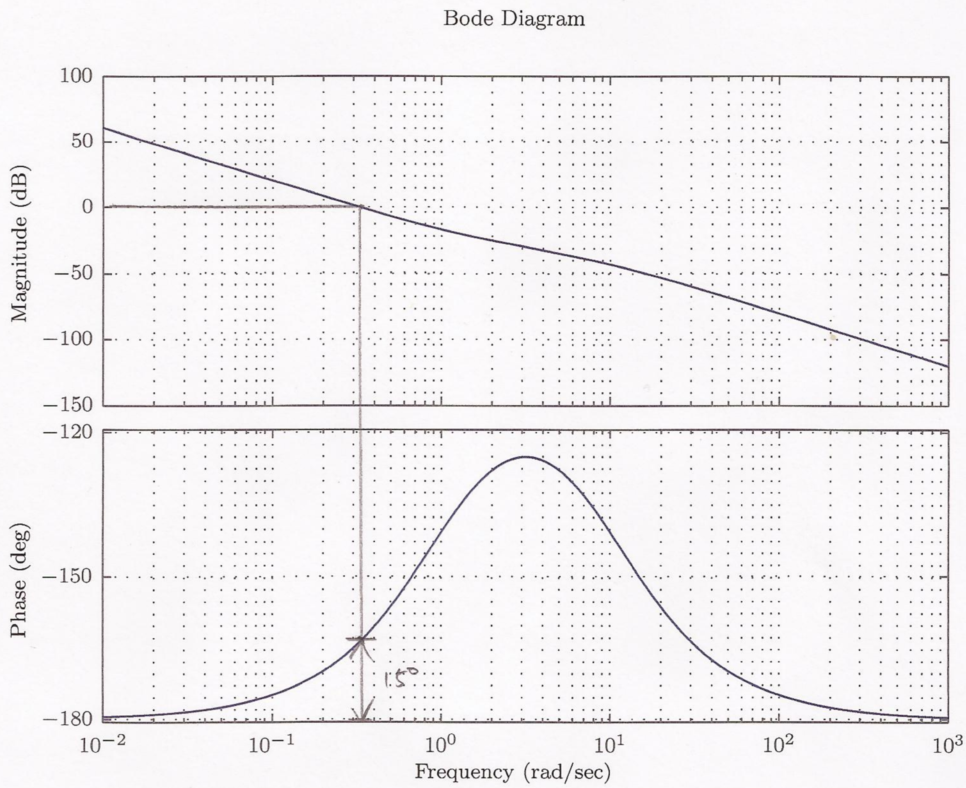
The phase decreases by 270° even though there are only two poles, which means the zero must be in the RHP. Step response goes "the other way" initially, suggesting a RHP zero.

Problem 3 (30 points)

Consider a system comprised of a lead compensator and gain applied to a plant with n poles at the origin as shown in the figure below:



The Bode plot of the open-loop system is given in the figure below:



(a) From the Bode plot, what is the value of n for this plant? Explain how you know this.

The magnitude drops by 40 dB per decade at low frequencies $\Rightarrow n=2$

OR

The phase plots starts at $-180° \Rightarrow n=2$

(b) If you know that this plot corresponds to a gain value $K = 1$ and a zero location of $s = -1$, where is the pole located? What are the values of α and T used to generate the lead compensator?

From the Bode plot, the pole is at $s=-10$

Zero: $s=-1$

$\Rightarrow Ts+1 = s+1 \Rightarrow T=1$

$\alpha s+1 = 0.1s+1 \Rightarrow \alpha = 0.1$

$$\Rightarrow D(s) = \frac{s+1}{0.1s+1}$$

(c) What is the phase margin with the current gain of 1? Show on the plot how you find the phase margin.

PM $\approx 15°$; look at where the magnitude equals 1

(d) What is the maximum phase margin you could achieve by changing the gain? Approximately what value of gain would you need? Explain. What would be the best damping ratio you could achieve this way?

$-30\,dB = 20\log(|G|)$

$|G| = 10^{-1.5} = \frac{1}{\sqrt{1000}} \approx \frac{1}{31}$

The peak of the phase plot is about $-125°$

$\Rightarrow \phi_{max} = -125° - (-180°) = 55°$

The magnitude is at $-30\,dB \Rightarrow$ need $K \approx 31$ for this

$$\zeta \approx \frac{PM}{100} = \frac{55°}{100} = 0.55$$

(e) What is the maximum phase margin you could achieve by changing T? Approximately what value would you need to achieve it? Explain.

For $\alpha = 0.1$, $1/\alpha = 10 \Rightarrow$ From Fig. 6.54, $\phi_{max} = 55°$

↳ was attached to the final exam

Center the lead at the point where the magnitude equals 1

$\Rightarrow$ pick a zero at $s=-0.1$ and a pole at $s=-1$

$\Rightarrow T=10$

(f) What system type is the closed-loop system? For which inputs will there be zero steady-state error?

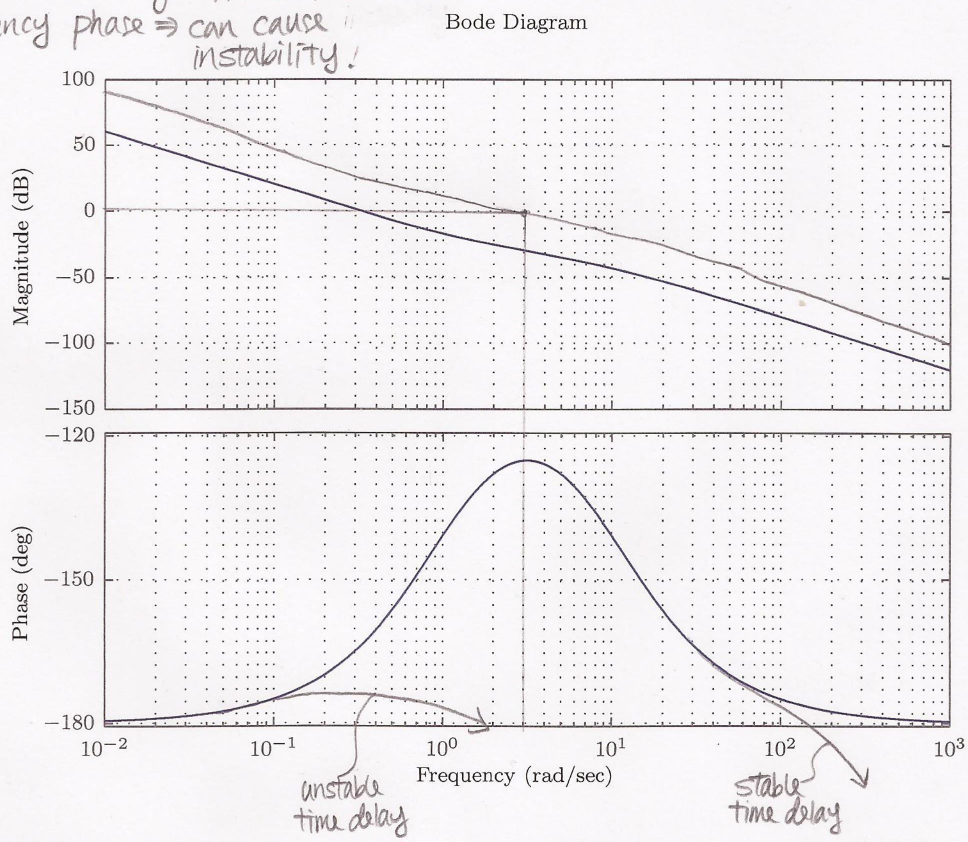
Can factor out 2 poles at the origin in the open-loop TF

⇒ Type 2 system

⇒ For step and ramp inputs, there will be zero steady-state error

(g) Suppose you choose the gain K from part (d) to close the feedback loop but you are concerned about delay in the system. On the Bode plot below, draw your open-loop system with the gain K from part (d) and the effects of two different delay times – one for which the system is stable and one for which it goes unstable. Explain your reasoning.

Large time delay affects low frequency phase ⇒ can cause instability!



(h) Suppose you had derivative feedback available and decided to replace the lead compensator with a PD controller. What values of proportional and derivative gain would you need so that the closed-loop poles had the damping ration calculated in part (d) and a natural frequency of 4 rad/s?

$\zeta = 0.55 \rightarrow$ say $\zeta = 0.5$

$\omega_n = 4$ rad/s

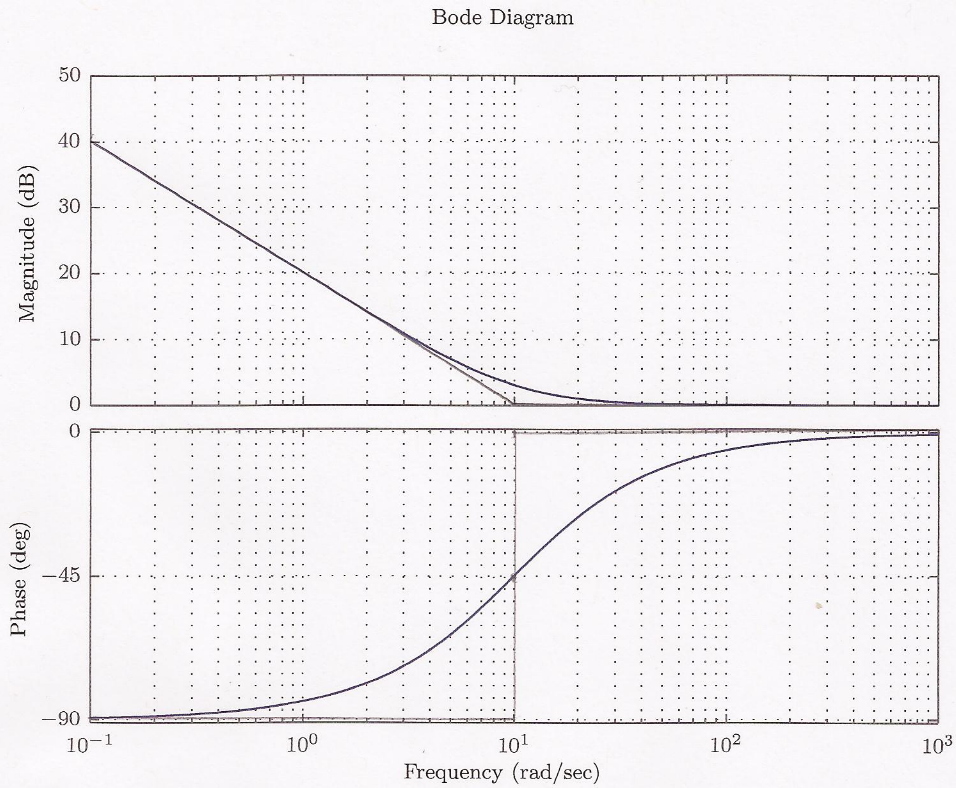
$$CL \Rightarrow \frac{DG}{1+DG} = \frac{0.1(k_P + k_D s)}{s^2 + \frac{k_D}{10}s + \frac{k_P}{10}}$$

$$\frac{k_P}{10} = \omega_n^2 = 16 \Rightarrow k_P = 160$$

$$\frac{k_D}{10} = 2\zeta\omega_n = 4 \Rightarrow k_D = 40$$

Problem 4 (10 points)

Solve for the step response of the system described by the Bode plot below to a step input, assuming zero initial conditions. The system is described by nice, round numbers.



1 pole at the origin, 1 zero at $s=-10 \Rightarrow G(s) = \frac{s+10}{s}$

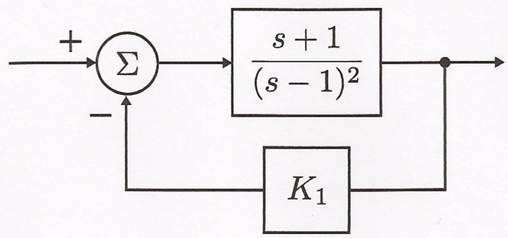
$\frac{Y(s)}{U(s)} = G(s) = \frac{s+10}{s}, \quad U(s) = \frac{1}{s}$

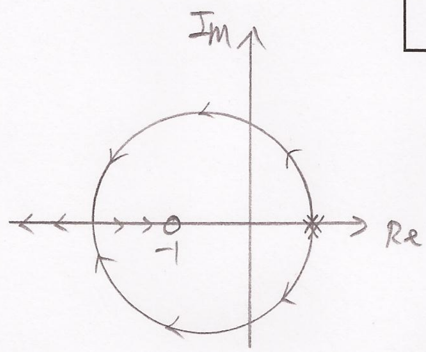
$\Rightarrow Y(s) = \frac{s+10}{s^2} = \frac{1}{s} + \frac{10}{s^2}$

$y(t) = \mathcal{L}^{-1}[Y(s)] = (1+10t)\,1(t)$

Problem 5 (15 points)

(a) Suppose we want to stabilize the system below by using gain K_1. Draw a root locus for this system.





(b) For what range of K_1 is the system stable?

Characteristic eq. $\Rightarrow 1 + K_1 \frac{s+1}{(s-1)^2} = 0$

$\Rightarrow (s-1)^2 + K_1(s+1) = 0$

$s^2 - 2s + 1 + K_1 s + K_1 = 0$

$s^2 - (2-K_1)s + (1+K_1) = 0$

Row 2:	1	$1+K_1$
1:	K_1-2	0
0:	$1+K_1$	

$\Rightarrow$ need $K_1 > 2$ to be stable

(c) What gain K_1 is necessary so that the poles of the system have a natural frequency of 3 rad/s? What is the corresponding damping ratio? (If you get a number that isn't nice looking for the damping ratio, go back and check things...)

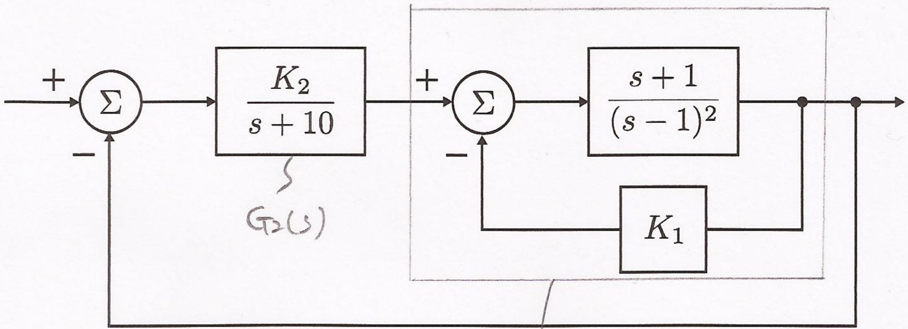
$\omega_n = 3$ rad/s

$s^2 + (K_1-2)s + (1+K_1) = 0$

$1 + K_1 = \omega_n^2 = 9 \Rightarrow K_1 = 8$

$2\zeta\omega_n = K_1 - 2 = 6 \Rightarrow 6\zeta = 6 \Rightarrow \zeta = 1$

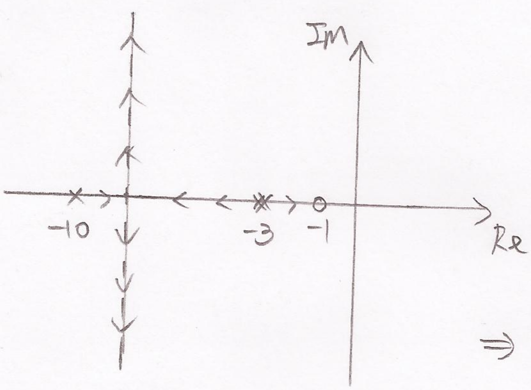
(d) Suppose we now close a feedback system around this entire system with some additional actuator dynamics. Draw a root locus for the combined system when K_1 is set as you calculated it. Determine the range of gain K_2 for which the combined system is stable.



$$G_1(s) = \frac{\frac{s+1}{(s-1)^2}}{1 + K_1 \frac{s+1}{(s-1)^2}} = \frac{s+1}{(s-1)^2 + K_1(s+1)}$$

$$G(s) = G_1(s)\,G_2(s) = \frac{k_2(s+1)}{[(s-1)^2 + k_1(s+1)](s+10)} = \frac{k_2(s+1)}{[s^2 + (k_1-2)s + (k_1+1)](s+10)}$$

As calculated, $G(s) = \dfrac{k_2(s+1)}{(s^2+6s+9)(s+10)} = \dfrac{k_2(s+1)}{(s+3)^2(s+10)}$



$n-m = 2 \Rightarrow$ asymptote $= \pm 90°$

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{(-6-10)-(-1)}{2} = -7.5$$

$\Rightarrow$ The system is stable for all $k_2 > 0$