Assignment 5 Solutions

Problem 1)

a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$

Routh array/table

s^4 : $1 \quad 32 \quad 100$

s^3 : $8 \quad 80$

s^2 : $\dfrac{32\times 8 - 80}{8} = 22 \quad \dfrac{8\times 100}{8} = 100$

s^1 : $\dfrac{80\times 22 - 8\times 100}{22} = 43.6$

s^0 100

all terms positive, no sign change

$\Rightarrow$ no roots in the RHP

b) $s^4 + 6s^2 + 25 = 0$

Two co-effs are missing. So there are roots outside LHP

create a new row by $\dfrac{d\,a(s)}{ds} = 4s^3 + 12s$

Routh array:

s^4: $1 \quad 6 \quad 25$

s^3: $4 \quad 12$

s^2: $\dfrac{6\times 4 - 12}{4} = 3 \quad \dfrac{25\times 4}{4} = 25$

s^1 : $12 - \dfrac{100}{3} = -21.3$

s^0 : 25

2 sign changes

$\Rightarrow$ 2 roots not in LHP

Problem 2)

characteristic eqn:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

construct routh array

$$\begin{array}{llll} s^5: & 1 & 10 & 5 \\ s^4: & 5 & 10 & K \\ s^3: & a_1 & a_2 & \\ s^2: & b_1 & K & \\ s^1: & c_1 & & \\ s^0: & K & & \end{array}$$

$$a_1 = \frac{5(10) - 1(10)}{5} = 8 \qquad a_2 = \frac{5(5) - 1(k)}{5} = \frac{25-K}{5}$$

$$b_1 = \frac{a_1(10) - 5(a_2)}{a_1} = 10 - \frac{25-k}{8} = \frac{55+K}{8}$$

$$c_1 = \frac{b_1(a_2) - a_1(K)}{b_1} = \frac{\left(\frac{1375 - 30K - K^2}{40} - 8K\right)}{\left(\frac{55+K}{8}\right)} = \frac{-(K^2 + 350K - 1375)}{5(55+K)}$$

For stability all elements in first column should be positive

$\Rightarrow$ 1 $b_1 = \frac{55+k}{8} > 0 \Rightarrow k > (-55)$

2 $c_1 = \frac{-(K^2+350K-1375)}{5(55+K)} > 0$ (note $K > -55$ from 1 so we can multiply)

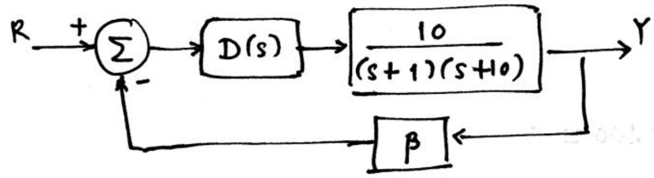
$$\Rightarrow -(K^2 + 350K - 1375) > 0$$

$$\Rightarrow -(K - 3.885)(K + 353.885) > 0 \quad \& \quad K > -55$$

$$\Rightarrow -55 < K < 3.885$$

3 $d_1 = K > 0$

Problem 3)



a) Need an integrator but also need to make it stable so chose $D_2(s)$

$$T(s) = \frac{Y(s)}{R(s)}$$

$$e_{ss} = \lim_{s \to 0} s \left[\frac{s(s+1)(s+10)}{s(s+1)(s+10) + 10(k_p s + k_I)} \right] \frac{1}{s^2}$$

$$= \frac{1}{k_I}$$

for (e_{ss} to ramp) $\lesssim \frac{1}{10}$

$\Rightarrow \boxed{k_I \geq 10}$ — (1)

closed loop poles are the roots of

$$s^3 + 11s^2 + 10s + 10(k_p s + k_I) = 0$$

apply routh criteria

s^3:	1	$10(1+k_p)$
s^2:	11	$10k_I$
s^1:	$\frac{110(1+k_p) - 10k_I}{11}$	
s^0:	$10k_I$	

$\Rightarrow$ $k_I > 0$ & $11(1+k_p) - k_I > 0$ for stability

$\Rightarrow 11(1+k_p) > k_I > 10$ — (1) from (1)

$\Rightarrow k_p > \frac{10}{11} - 1 = \left(\frac{-1}{11}\right)$ Choose $k_p = 0$

for 0.01

$k_I \geq 100$

$11(1+k_p) > k_I \geq 100$

$\Rightarrow (1+k_p) > \frac{100}{11}$

$\Rightarrow k_p > \frac{89}{11} = 8.09$

for $\boxed{\beta = 0.9}$,

b) $$E(s) = \frac{s(s+1)(s+10) - k_p - k_I}{s(s+1)(s+10) + 9(k_p s + k_I)} R(s)$$

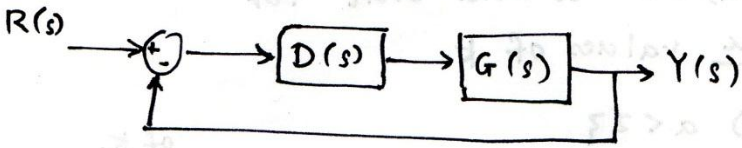
$e_{ss,\,ramp} \rightarrow \infty$

system is no longer type I

Problem 4)

second order system $G(s) = \dfrac{1}{s^2 + 2\zeta s + 1}$

$$D(s) = K\frac{(s+a)}{(s+b)}$$



a) $$\frac{Y(s)}{R(s)} = \frac{DG}{1+DG} = \frac{K(s+a)}{(s^2+2\zeta s+1)(s+b)+k(s+a)}$$

The error relation

$$\frac{E(s)}{R(s)} = \frac{1}{1+DG}$$

for Type I there needs to be a pole at $s=0$ in the term 'DG'

There is no such pole in G

$\therefore$ For type I, $(s+b) = s$ ie. $\underline{b=0}$

b) characteristic eqn with $b=0$ is

$$(s^2+2\zeta s+1)s + k(s+a) = 0$$

routh array

s^3: 1 $\quad 1+k$

s^2: 2ζ $\quad ak$

s^1: $\dfrac{2\zeta(1+k)-ak}{2\zeta}$

s^0: aK

requirements for stability

$$\left(\begin{array}{c} 2\zeta(1+K) - aK > 0 \\ aK > 0 \end{array}\right)$$

c) Assume $\zeta > 0$

We are analyzing for all positive k $(k > 0)$

$\Rightarrow$ for $ak > 0$ to hold $a > 0$

for $2\zeta + k(2\zeta - a) > 0$ to hold true for

<u>all positive</u> values of k

$2\zeta - a > 0 \Rightarrow a < 2\zeta$

(if $a < 2\zeta$ then for any positive value of k it will be stable)

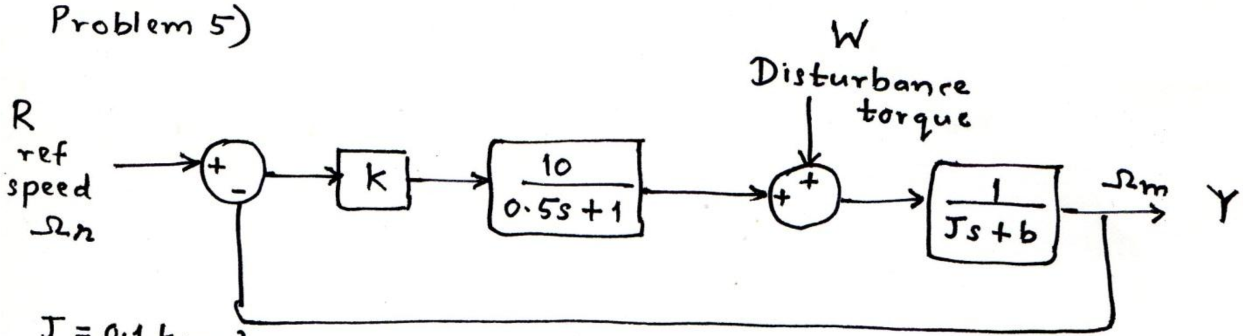
if $a > 2\zeta$ then $(a - 2\zeta) > 0$

and $2\zeta + k(2\zeta - a) > 0$

$\Rightarrow 2\zeta > k(a - 2\zeta)$

$\Rightarrow$ $0 < k < \dfrac{2\zeta}{a - 2\zeta}$ is the range of k for which the system is stable if $a > 2\zeta$

Problem 5)



$J = 0.1$ kg-m^2

$b = 1$ N-m-sec

a) TF for disturbance

$$\frac{Y}{W} = \frac{\left(\frac{1}{Js+b}\right)}{1+\left(\frac{1}{Js+b}\right)\left(\frac{10K}{0.5s+1}\right)} \qquad b=1 \quad J=0.1$$

$$= \frac{(0.5s+1)}{(0.1s+1)(0.5s+1)+10K}$$

$$e_{ss}\ (\text{step in } W) = \lim_{s\to 0} s \cdot \frac{1}{s}\left(\frac{Y}{W}\right) = \frac{1}{1+10K}$$

$W(s) = \frac{1}{s}$

for $e_{ss} \leq 0.01 \Rightarrow \frac{1}{1+10K} \leq 0.01$

$\Rightarrow 10K \geq 99$

$\Rightarrow K \geq 9.9$ (pick $K = 10$ say)

b) The roots of any CLTF will be those of $1+DG=0$ (irrespective of the disturbance)

Characteristic eqn is

$$(0.1s+1)(0.5s+1) + 10K = 0 \quad (K=10)$$

$$0.05s^2 + 0.6s + (1+100) = 0$$

$$\Rightarrow s^2 + 12s + 2020 = 0$$

roots are $-6 \pm 44.542i$

find the second order parameters

$\omega_n = \sqrt{2020} \simeq 45$

$\zeta = \frac{12}{2\sqrt{2020}} \simeq 0.13$

$\Rightarrow$ damping is too low, would result in a high overshoot

Overshoot $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.6547$

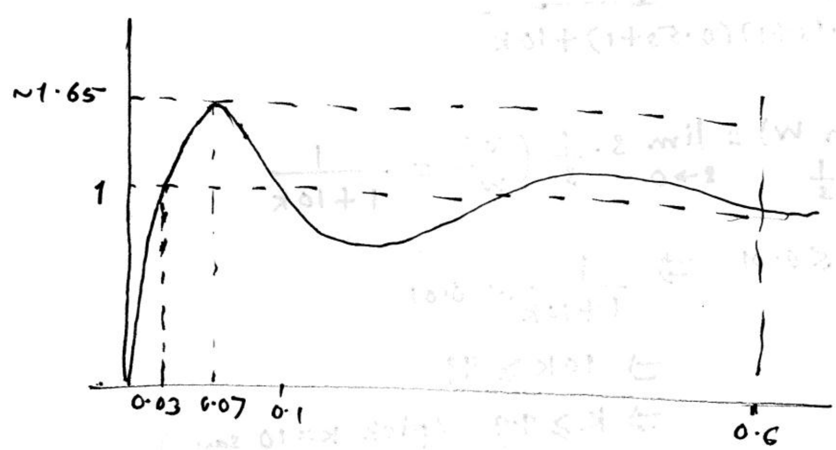
$\Rightarrow$ 65.47 % overshoot

Rise time $\simeq$ 0.03 sec

settling time $\simeq$ 0.65 sec

peak time $\simeq$ 0.07 sec

c) Rough sketch



Problem 5) contd..

d) $t_s \leq 0.1$ sec and $M_p \leq 5\%$

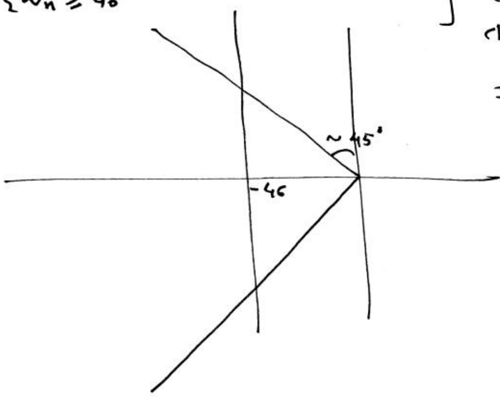
$\Rightarrow \sigma \geq 46$ $\qquad \Rightarrow \zeta \geq 0.7$

$\zeta \omega_n \geq 46$

combining those two, chose $\zeta = 0.7$ (say)

$\Rightarrow \omega_n \geq 65.71$

$\omega_n = 66$ (say)



e) Method I) Replace K by $K_p + K_d s$

Method II) Do not change K just add $\boxed{\text{PD}}$ in feedback

Note – Method 1 does not give us the normal second order system and you should approximate it as one. Method 2 gives you a second order system. Note that you have to add (Kp+Kd*s) in feedback and still keep the K that you have in the main loop.

$$\frac{Y(s)}{R(s)} = \frac{200(k_p + k_d s)}{\left\{ s^2 + \cancel{600}\,(12 + 200 k_d) s + (200 k_p + 20) \right\}}$$

Desired $\zeta = 0.7$, $\omega_n = 66$

Approx. by second order response. look at the den. (poles) which will govern transients

$\sqrt{200 k_p + 20} = \omega_n = 66$

$\Rightarrow \boxed{k_p = 21.68}$

$12 + 200 k_d = 2(0.7)(66) = 92.4$

$\Rightarrow \boxed{k_d = 0.402}$

Part d Method 2:

Add a PD block in feedback

$(R - (K_p + K_d s) Y) \, KDG = Y$

$\Rightarrow \cancel{\frac{R}{Y}} \quad \Rightarrow \frac{Y}{R} = \frac{KDG}{1 + KDG(K_p + K_d s)}$

$$= \frac{10K \times 20}{(s+2)(s+10) + 10K(K_p + K_d s) \times 20}$$

$\omega_n^2 = 20 + 200K K_p$

$2\zeta\omega_n = 12 + 200K K_d$.

choose appropriate values

f) Disturbance induced steady state error will be a constant $\neq 0$

To correct this integral term $\left(\frac{K_I}{s}\right)$ needs to be added to the controller