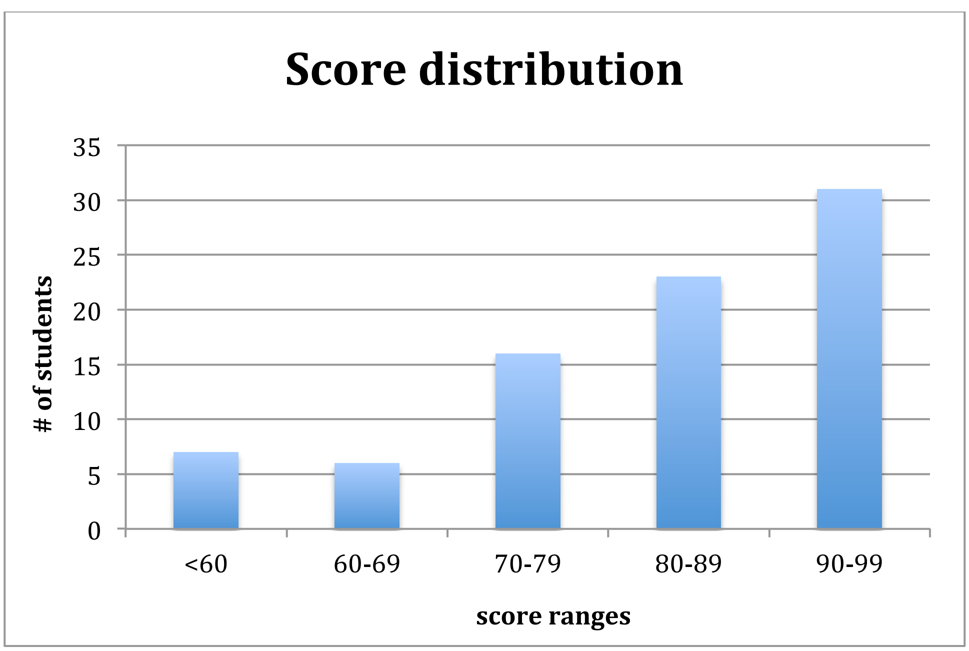
ENGR 105: Feedback Control Design
Final Exam, Winter Quarter 2013
Wednesday, March 20, 2013, 8:30-11:30 am, Room 320-105

SOLUTIONS

Problem	Mean / Std. Dev.
1 (10 pts.)	8.3 / 1.5
2 (25 pts.)	21.4 / 2.8
3 (25 pts.)	21.0 / 3.9
4 (15 pts.)	12.2 / 3.3
5 (25 pts.)	18.9 / 5.4
Total (100 pts.)	81.8 / 12.8



Problem 1. (10 pts.)

Mark the following True (T) or False (F):

__T__ a. (1 pt.) It is impossible in the real world to create a perfect proportional-derivative controller.
not possible

__F__ b. (1 pt.) A lead controller resembles a proportional-derivative controller at high frequencies.
low

__T__ c. (1 pt.) One can always find a value of feedback gain K that will make a non-minimum phase system unstable. (since at least one pole must go to RHP)

__T__ d. (1 pt.) The 0° root locus is useful for studying the behavior of systems with negative feedback gain ($K < 0$).

__F__ e. (1 pt.) You can always tell the stability of a system by looking at the Nyquist plot produced by the MATLAB command `nyquist`. (you cannot see loops at ∞)

__F__ f. (1 pt.) A Type 1 system, which has infinite error to a parabola input, is unstable.
not a BIBO situation → type does not impact stability

Circle the correct answer for the following questions:

g. (1 pt.) Does the addition of a zero to a system tend to make the system **MORE** or **LESS** responsive to changes in input?

h. (1 pt.) Consider a stable transfer function $G_1(s) = \frac{s+1}{(s+2)(s+3)}$ and an unstable transfer function $G_2(s) = \frac{s+1}{(s-2)(s+3)}$. Do they have the same magnitude Bode plot? **YES** or **NO**

Provide a short narrative answer to each of the following questions:

i. (1 pt.) What is a physical interpretation of the zeros of a guitar?

The zeros of a guitar are related to how the system is forced → the way the strings are touched excites particular modes of vibration.

j. (1 pt.) Why might you want to determine stability from a Nyquist plot rather than the Bode plot?

There are ~~restrict~~ restrictive conditions on when you can use GM & PM read from a Bode plot.

Nyquist always works.

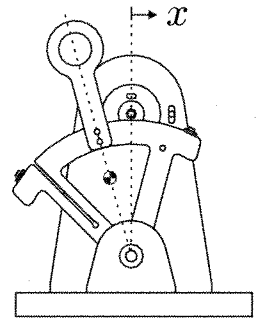
(some other answers also acceptable)

Problem 2. (25 pts.)

In the laboratory, you worked with a one-degree-of-freedom inverted pendulum whose equation of motion can be simplified to the version shown below.

$$m\ddot{x} + b\dot{x} - kx = f$$

where m, b, and k are all > 0



a. (1 pt.) What is the physical reason for the negative sign in front of k?

gravity pulls the handle away from $x = 0$

The pendulum is inverted

b. (1 pt.) What is the transfer function $G(s)$ of this device (the open-loop system)?

$$ms^2 X(s) + bs X(s) - k X(s) = F(s)$$

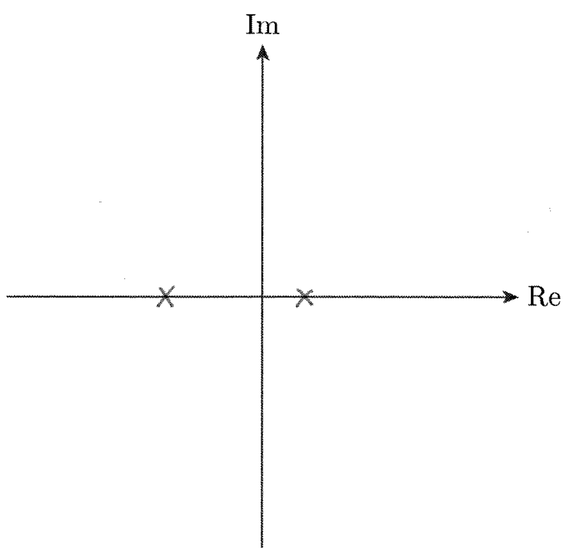
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs - k}$$

c. (3 pts.) According to the transfer function, is the device stable? Give any poles and zeros of the system, with the real and imaginary parts defined in terms of the given parameters, and plot them on the s-plane.

Is it stable? No

List of any poles and zeros: $s_{1,2} = -\frac{b}{2m} \pm \frac{\sqrt{b^2 + 4mk}}{2m}$ (no zeros)

s-plane:



both poles are real

since $\sqrt{b^2 + 4mk} > b$,

there will be

a positive real pole.

d. (5 pts.) Now we will consider this device as the plant in a unity negative feedback control system, where the controller is $D(s) = 2k$ and $m = b = k = 1$. This closed-loop system is subjected to unit reference step input. Give the values of the following time-domain response metrics? (Note that you can refer to the table on the last page. Another useful calculation is $\frac{\pi}{\sqrt{0.75}} = 3.6$)

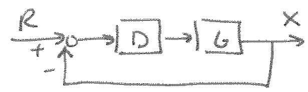
$M_p =$ 16 %
% maximum overshoot

$t_p =$ 3.6 sec
peak time (sec)

$t_r =$ 1.8 sec
approximate 10-90% rise time (sec)

$t_s =$ 9.2 sec
1% settling time (sec)

Since this is a C.L. 2nd order system, we can use metrics derived in class



$\frac{X}{R} = \frac{DG}{1+DG} = \frac{2k}{ms^2+bs+k}$

$\frac{X}{R} = \frac{2}{s^2+s+1} = \frac{2\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $\therefore$ $\omega_n = 1$, $\zeta = 0.5$, $\sigma = \zeta\omega_n = 0.5$

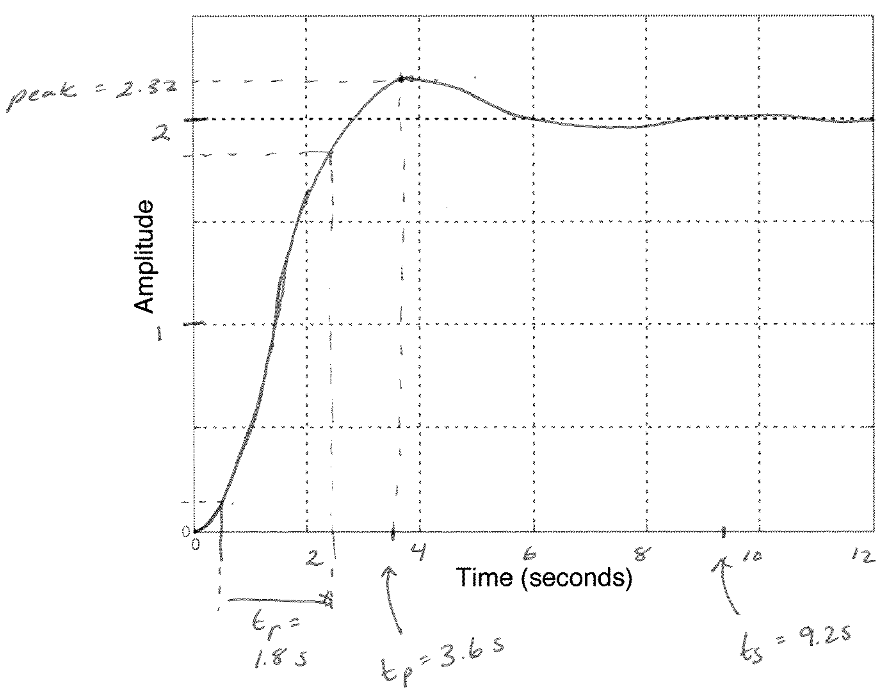
M_p for $\zeta = 0.5 \rightarrow 16\%$ (table)

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{1 \cdot \sqrt{0.75}} = 3.6$ sec

$t_r \approx \frac{1.8}{\omega_n} = 1.8$ sec $t_s = \frac{4.6}{\sigma} = 9.2$ sec

e. (5 pts.) Draw the time domain response to a unit step reference for the case in part d. Compute any additional metrics needed to develop an accurate drawing.

Additional metric(s) and their values: final value = 2 other possible: $e_{ss} = 1$, $\omega_d = \sqrt{0.75}$



Final value:

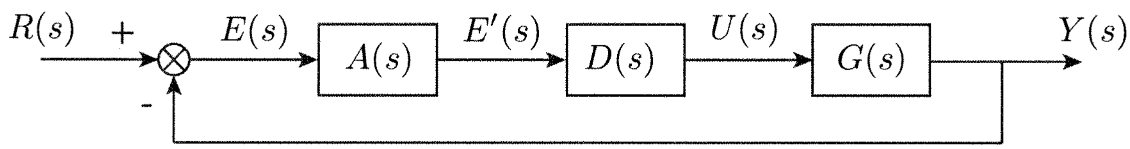
$\lim_{s \to 0} s \cdot \frac{1}{s} \frac{2k}{ms^2+bs+k} = 2$

$M_p = 16\%$

$\therefore$ peak is $2 + 2(0.16) = 2.32$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$
$= \sqrt{0.75}$ rad/s

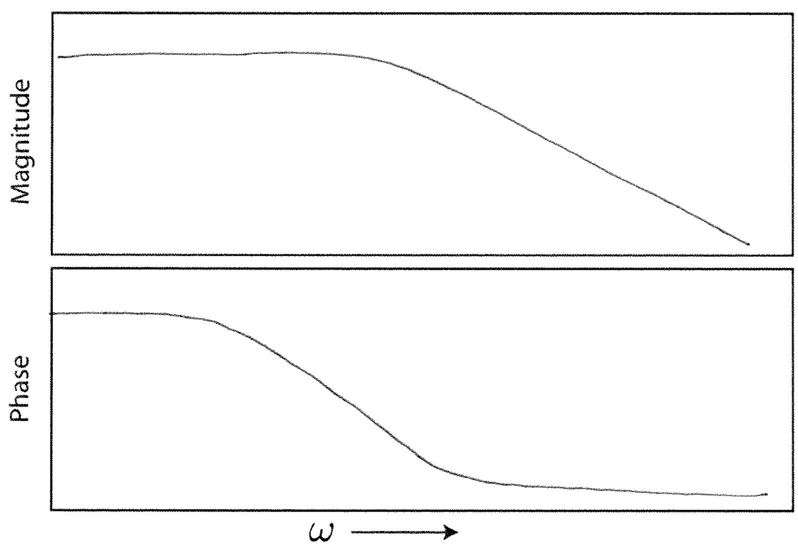
f. (5 pts.) Now consider the use of a PD controller, $D(s) = K_p + K_d s$. As we saw in lab, the position/velocity signal was filtered to remove position measurement noise from the error signal. The block diagram below shows a system with such a filter included, where $A(s)$ is the filer and $E'(s)$ is the filtered error signal.



What is the simplest form of $A(s)$ that could be used to remove high-frequency noise from the error signal?

$A(s) = \frac{a}{s+a}$ (or $\frac{1}{s+a}$)   * do. not want $\frac{1}{s}$, since it will impact low freq.

Roughly sketch the Bode plot of $A(s)$. The Bode plot does not need any units on the axes – you only need to show the shapes of the curves.



When $A(s)$ and $D(s)$ are combined to become $A(s)D(s)$, what standard controller do they resemble?

lead compensator

$\frac{1}{s+a} \cdot k_p + k_d s$

$= \frac{k_p/k_d + s}{s + a} k_d$

↑ same form as lead compensator

g. (5 pts.) Now we wish to use a different controller to create a closed-loop system that:

- is stable
- has no steady-state error to a step ← integral control
- meets a settling time specification
- meets a peak time specification

$\Big\}$ proportional - derivative

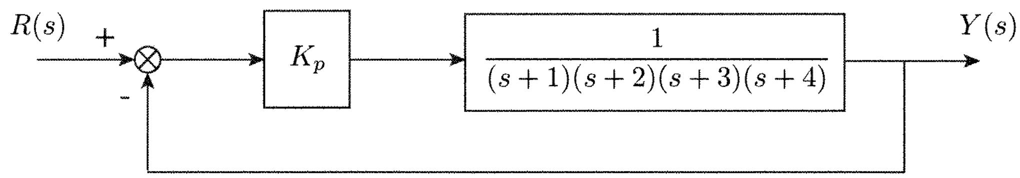
What type of controller would satisfy these requirements? PID $\quad k_p + k_d s + \frac{1}{s} k_i$

With this controller, would you be able to use the equations for the time-domain system response metrics that you used in part d? Explain why or why not:

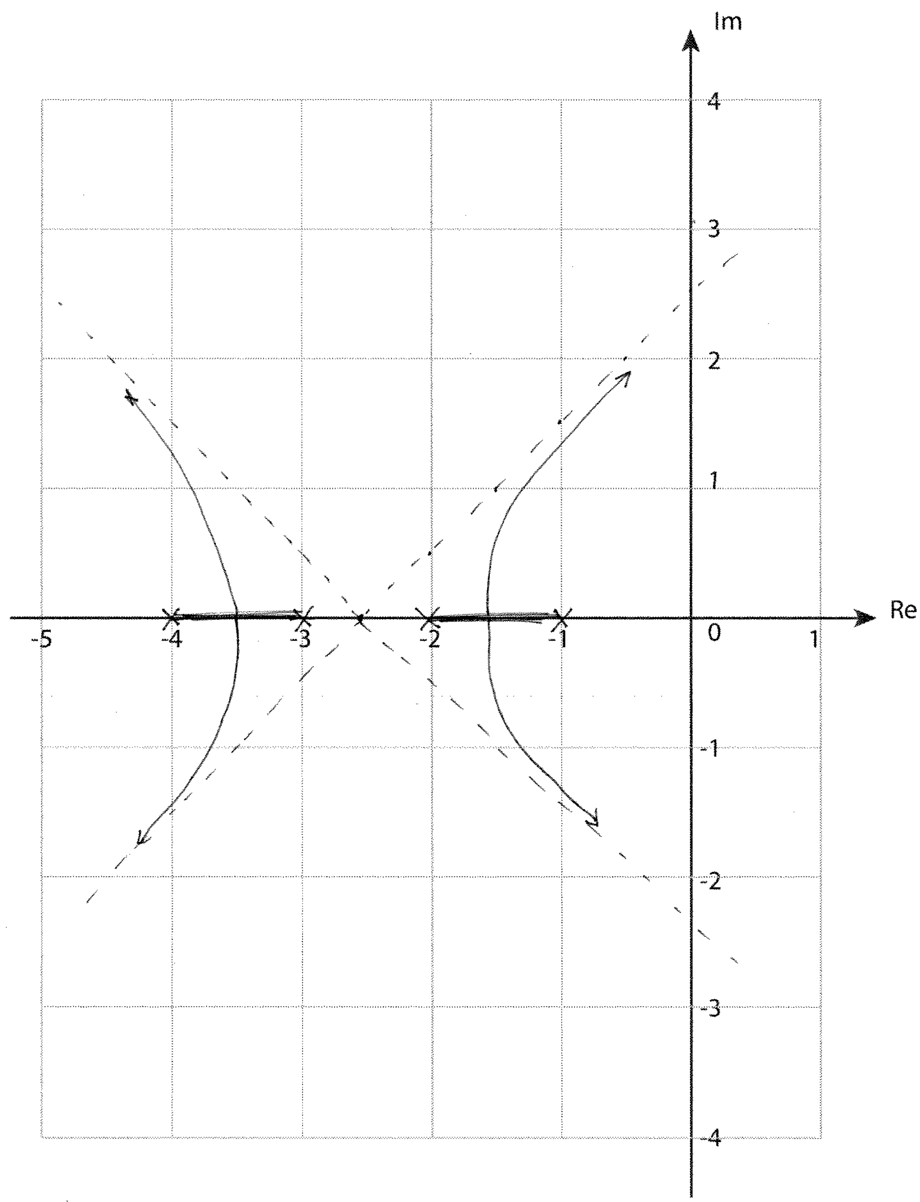
No. The closed-loop system is no longer 2nd order (w/ no zeros), so the same equations for the metrics would not apply.

Problem 3. (25 pts.)

Consider the closed-loop system shown in the block diagram below.



a. (1 pt.) Place any poles and zeros of the open-loop system (the plant) on the s-plane below.



b. (9 pts.) Calculate the following guidelines for sketching the root locus $1 + K_p G(s)$, and draw asymptotes on the s-plane in part a.

The angles of the asymptotes are: $45°, 135°, 225°, 315°$

The centers of the asymptotes are: -2.5

The departures from the real axis occur at the angles: $\pm 90°$

(I am not asking you calculate the locations of departure from the real axis. You can approximate those when you sketch the root locus in the next part.)

angles of asymptotes: $\dfrac{180° + 360°(l-1)}{4} = 45°, 135°, 225°, 315°$

centers of asymptotes: $\dfrac{\Sigma p - \Sigma z}{4} = \dfrac{-1-2-3-4}{4} = \dfrac{-10}{4} = -2.5$

departure angles from real axis: always $\pm 90°$ for 2 poles

c. (3 pts.) Sketch the asymptotes and the root locus on the s-plane on the previous page, using your answers to part b as a guide.

d. (5 pts.) Compute the values of K_p (consider only positive values) for which the system is stable.

K_p < 126

$1 + K_p G(s) = 0$

$$k_p + (s+1)(s+2)(s+3)(s+4) = (s^2+3s+2)(s^2+7s+12)$$
$$= s^4 + 10s^3 + 35s^2 + 50s + 24 + k_p$$

Routh array:

s^4	1	35	$24+k_p$
s^3	10	50	0
s^2	30	$24+k_p$	
s^1	$\frac{126-k_p}{3}$	0	
s^0	$24+k_p$		

$c_1 = \dfrac{-\det\begin{vmatrix} 1 & 35 \\ 10 & 50 \end{vmatrix}}{10} = 30$

$c_2 = \dfrac{-\det\begin{vmatrix} 1 & 24+k_p \\ 10 & 0 \end{vmatrix}}{10} = 24+k_p$

$d_1 = \dfrac{-\det\begin{vmatrix} 10 & 50 \\ 30 & 24+k_p \end{vmatrix}}{30} = \dfrac{-k_p+126}{3}$

$d_2 = 0$

$e_1 = \dfrac{-\det\begin{vmatrix} 30 & 24+k_p \\ \frac{126-k_p}{3} & 0 \end{vmatrix}}{\frac{126-k_p}{3}} = 24+k_p$

need $\dfrac{126-k_p}{3} > 0$

$\Rightarrow k_p < 126$

also $24 + k_p > 0$

$k_p > -24$

e. (7 pts.) At what value of ω is the system neutrally stable? Find the answer analytically; your answer should be exact. However, you do not need to simplify your answer (which might be difficult to calculate).

$$1 + K_p G = 0$$

$$1 + K_p \left(\frac{1}{s^4 + 10s^3 + 35s^2 + 50s + 24} \right) = 0$$

@ neutral stability, $K_p = 126$

$$\therefore \quad s^4 + 10s^3 + 35s^2 + 50s + 150 = 0$$

to find crossing, let $s = j\omega$ and solve for ω:

$$(j\omega)^4 + 10(j\omega)^3 + 35(j\omega) + 50(j\omega) + 150 = 0$$

$$\omega^4 - 10j\omega^3 - 35\omega^2 + 50j\omega + 150 = 0$$

real part

$$\omega^4 - 35\omega^2 + 150 = 0$$

$$\omega^2 = \frac{35}{2} \pm \frac{\sqrt{625}}{2}$$

$$= \frac{35 \pm 25}{2} = 30 \text{ or } 5$$

$$\therefore \quad \omega = \pm\sqrt{5} \text{ or } \sqrt{30}$$

imag. part

$$-10\omega^3 + 50\omega = 0$$

$$+10\omega^2 = 50$$

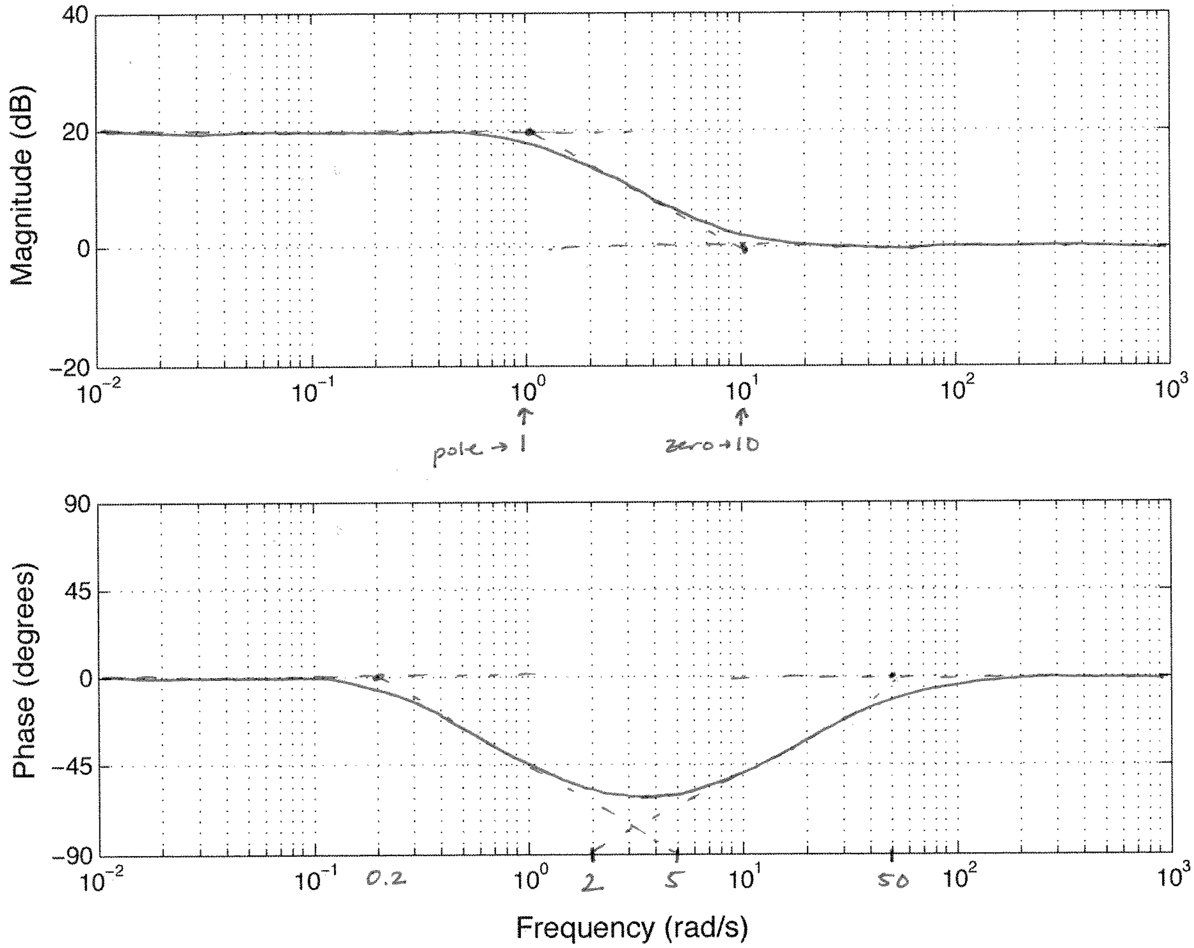
$$\omega = \pm\sqrt{5} \quad (\text{or } 0)$$

the matching answer (also can be seen from root locus sketch) is $\boxed{\omega = \pm\sqrt{5}}$

Problem 4. (15 pts.)

Consider a lag compensator of the form $D_{\text{lag}}(s) = K\frac{s+z}{s+p}$, where $z > p$, and a proportional-integral controller of the form $D_{\text{PI}}(s) = K_p + \frac{K_i}{s}$.

a. (5 pts.) For values of $z = 10, p = 1$, and $K = 1$, draw the Bode plot of the lag compensator $D_{\text{lag}}(s)$ on the grids below. Precisely draw the asymptotes for both the magnitude and phase plots, showing all necessary calculations. Then sketch the plots using the asymptotes as a guide.



$$D_{lag} = \frac{s+10}{s+1} = 10\,\frac{\left(\frac{1}{10}s+1\right)}{s+1}$$

Magnitude

$10 \to 20$ dB ~~/...~~ constant

zero → 0 until 10, 20 dB/dec after

pole → 0 until 1, −20 dB/dec after 10

Phase

$10 \to 0°$ phase

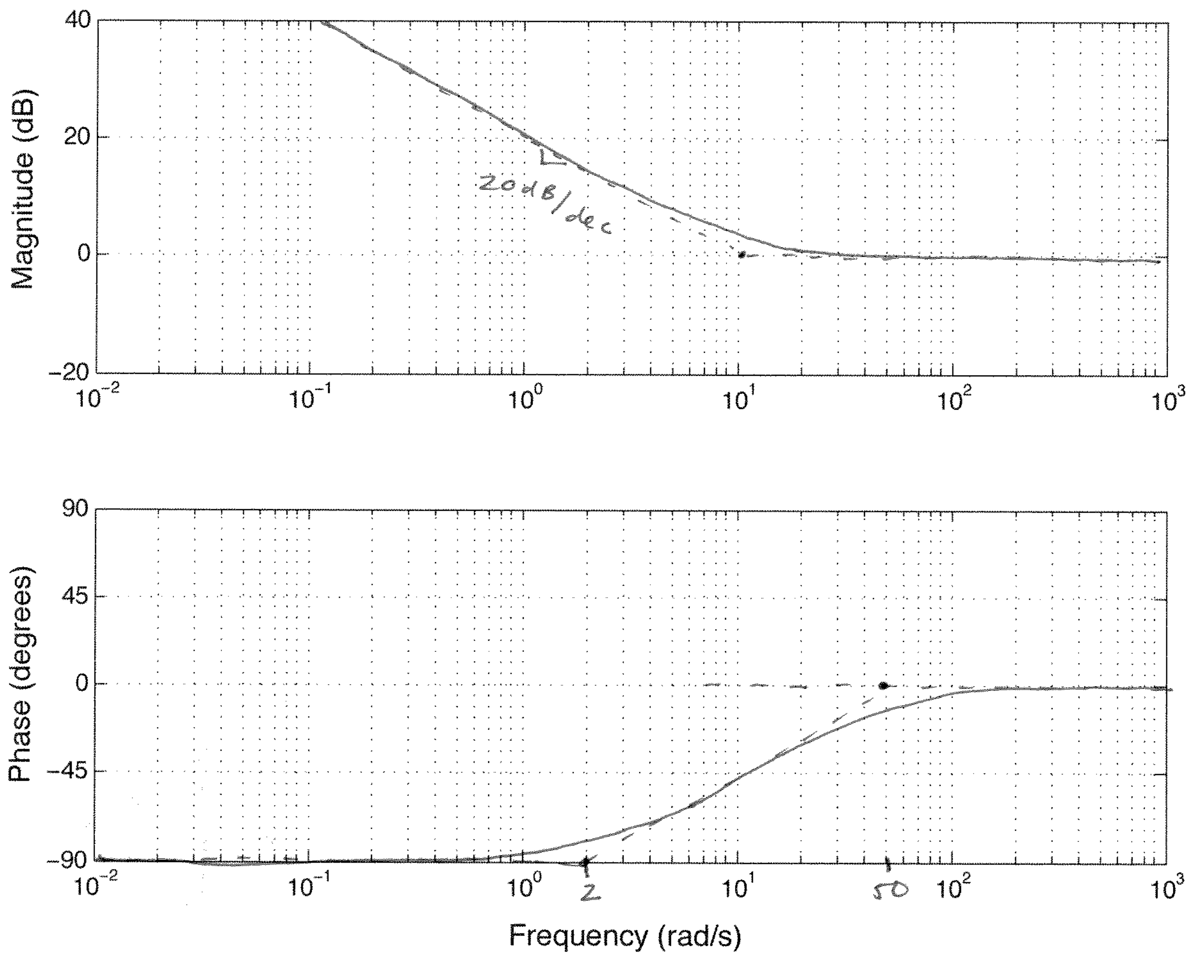
zero → $0° \to 90°$ between 2 & 50

pole → $0 \to -90°$ between 0.2 & 5

b. (10 pts.) Design a proportional-integral (PI) controller $D_{PI}(s)$ that closely matches the frequency response of the lag compensator above at **high** frequencies. Give the values of the PI gains and provide the Bode plot of the designed PI controller on the grids below. Show all your work.

K_p: 1

K_i: 2



$$D_{lag} = \frac{s+10}{s+1}$$

$$D_{PI} = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

this will be similar to D_{lag} if $K_p = 1$, $K_i = 10$

$$= \frac{s+10}{s} = 10\,\frac{\left(\frac{s}{10}+1\right)}{s}$$

mag:

pole$\left(\frac{1}{s}\right)$ is -20 dB/dec

zero is 0 until 10, then $+20$ dB/dec after

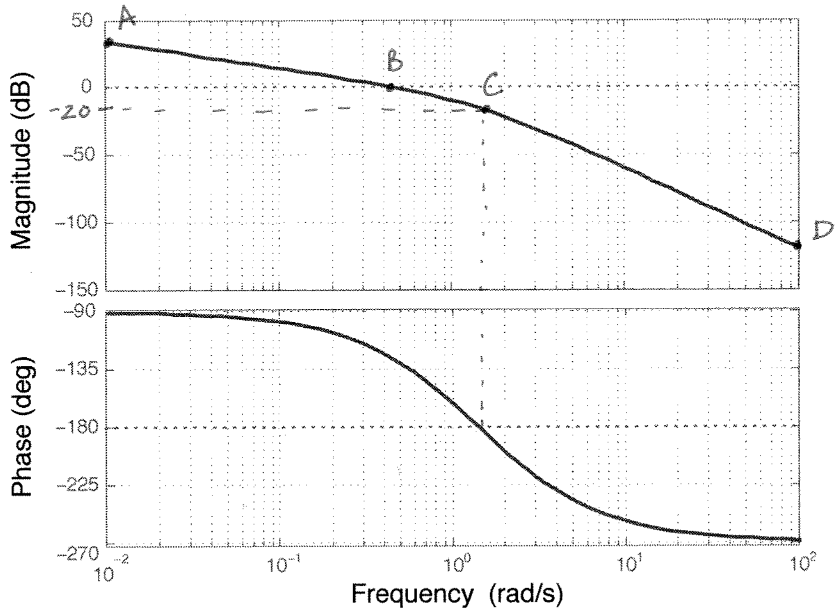
phase:

pole: $-90°$ always

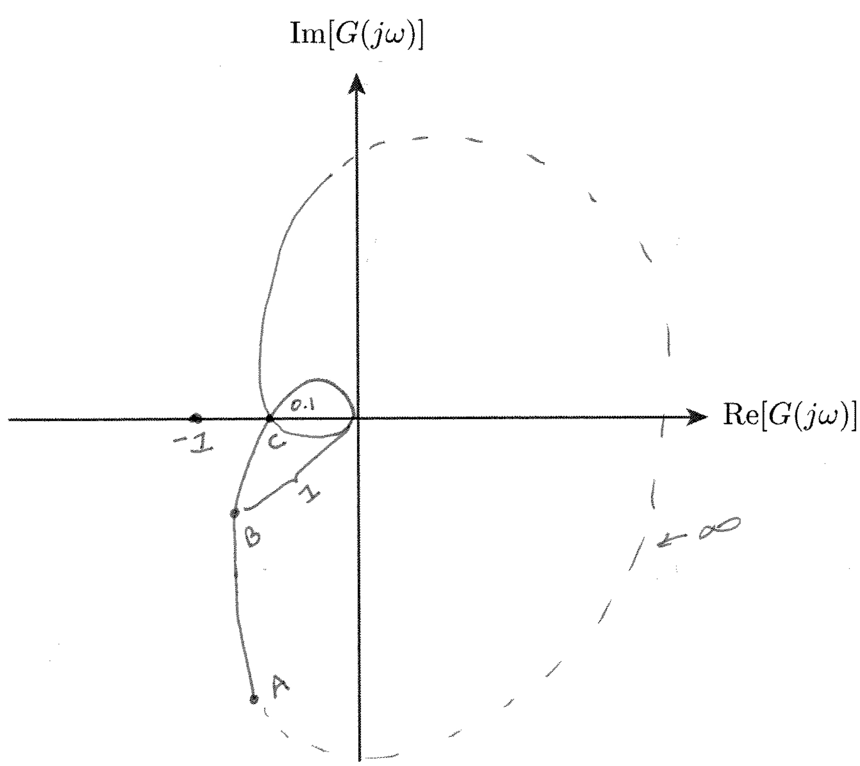
zero: 0–$90°$ from $2 \to 50$

Problem 5. (25 pts.)

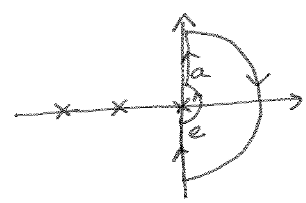
Below is a Bode plot for a system $G(s)$.



a. (5 pts.) Sketch the Nyquist plot for $G(s)$. Identify at least 4 important points (A, B, C, and D) on the Bode plot and show their corresponding positions as accurately as possible on the Nyquist plot. Include the -1 point on your Nyquist plot. Show any calculations needed to explain the behavior of any infinite points Nyquist plot. Draw arrows corresponding to the direction of the contour, and use a dashed line to indicate the parts of the plot at infinity.



@ $\omega = 0$



@ a, phase = $-90°$

@ e, phase = $90°$

test point in between, phase = $0°$

so line @ infinity crosses on right

also $-90° - 90° = -180°$, so we know it only goes round $\frac{1}{2}$ revolution

b. (10 pts.) Imagine that $G(s)$ is placed in a feedback system whose characteristic equation is $1 + KG(s)$. As K varies from 0 to infinity, describe the number of closed-loop right-half-plane poles and say whether the system is stable or unstable. Also, give the value(s) of K at which any transition in stability takes place. (Hint: You can return to the Bode plot to help with this.)

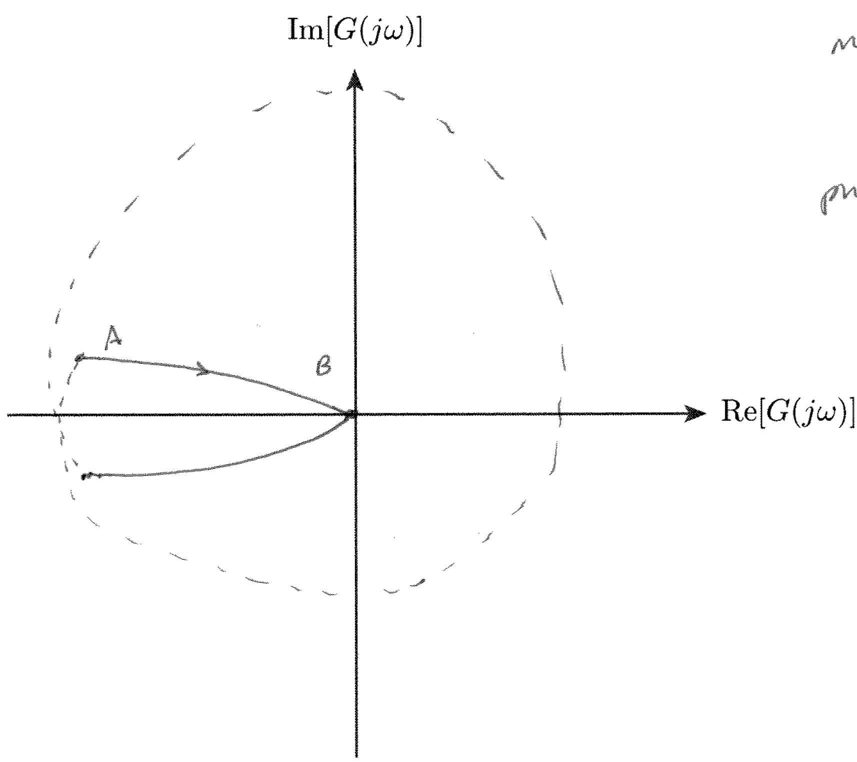
@ point C on plot, $-20\ \text{dB} = 0.1 = 1/K \rightarrow K = 10$

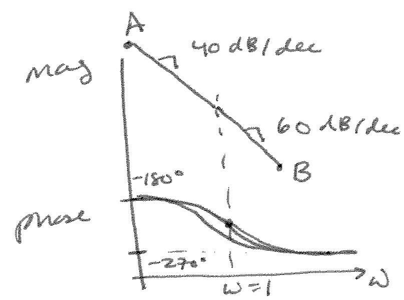
for $0 < K < 10$, no encirclements, stable, Ø RHP poles

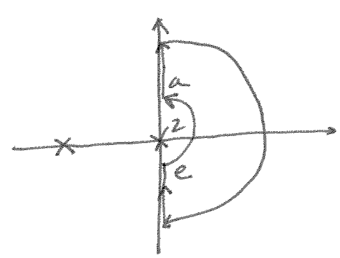
for $K > 10$, 2 CW encirclements, unstable, 2 RHP poles

c. (10 pts.) Now sketch the Nyquist plot for a different system, $G(s) = K\frac{1}{s^2(s+1)}$, for the case $K = 1$. Include the -1 point on your plot. Show any calculations needed to explain the behavior of any infinite points Nyquist plot. Draw arrows corresponding to the direction of the contour, and use a dashed line to indicate the parts of the plot at infinity.

For which values of $K > 0$ is the system unstable? all $K > 0$!







phase @ a = $-180°$
phase @ e = $180°$

$-180° - 180° = -360°$
so it goes round a whole revolution

Useful Tables

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t^2
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$\frac{1}{s+a}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{s^2+a^2}$	$\sin at$
18	$\frac{s}{s^2+a^2}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

Corresponding natural frequency and % maximum overshoot for unit step response of a second-order system:

ζ (zeta)	$e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$
0.1	0.73
0.2	0.53
0.3	0.37
0.4	0.25
0.5	0.16
0.6	0.095
0.7	0.046
0.8	0.015
0.9	0.002