Linear Optimization Problems

Read before the lecture

Chapter 6—Linear Programming

6.1 Standard LP Problem

6.2 Solution of Linear System of Equations

6.3 Basic Solutions of an LP Problem

6.4 The Simplex Method

6.5 Unusual Situations Arising During the Simplex Solution

After this lecture you should

Understand terms associated with linear programming

Be able to convert linear optimization problems to the standard LP Form

Understand the solution process for LP problems

Be able to compute solutions using the Simplex method

Standard LP Problem

Find $\boldsymbol{x}$ in order to

Minimize $\quad f(\boldsymbol{x}) \equiv \boldsymbol{c}^T \boldsymbol{x}$

Subject to $\quad \boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$.

where

$\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ vector of optimization variables

$\boldsymbol{c} = [c_1, c_2, \ldots, c_n]^T$ vector of objective or cost coefficients

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$ $m \times n$ matrix of constraint coefficients

$\boldsymbol{b} = [b_1, b_2, \ldots, b_m]^T \geq 0$ vector of right hand sides of constraints

Note that in this standard form the problem is of minimization type. All constraints are expressed as equalities with the positive right hand side. Furthermore all optimization variables are restricted to be positive.

Conversion to Standard LP Form

Maximization problem

Any maximization problem can be converted to a minimization problem simply by multiplying the objective function by a negative sign. For example

Maximize $z(\boldsymbol{x}) = 3x_1 + 5x_2$ is same as Minimize $f(\boldsymbol{x}) = -3x_1 - 5x_2$

Constant term in the objective function

Optimum solution $\boldsymbol{x}^*$ does not change if a constant is either added to or subtracted from the objective function. Thus a constant in the objective function can simply be ignored. After the solution is obtained, the optimum value of the objective function is adjusted to account for this constant.

Alternatively a new *dummy* optimization variable can be defined to multiply the constant and a constraint added to set the value of this variable to 1. For example consider the following objective function of two variables.

Minimize $f(\boldsymbol{x}) = 3x_1 + 5x_2 + 7$

In standard LP form it can be written as follows.

Minimize $f(\boldsymbol{x}) = 3x_1 + 5x_2 + 7x_3$

Subject to $x_3 = 1$

Negative values on the right hand sides of constraints

The standard form requires that all constraints must be arranged such that the constant term, if any, is a positive quantity on the right hand side. If a constant appears as negative on the right hand side of a given constraint, multiply the constraint by a negative sign. Keep in mind that the direction of inequality changes (that $\leq$ becomes $\geq$ and vice versa) when both sides are multiplied by a negative sign. For example

$3x_1 + 5x_2 \leq -7$ is same as $-3x_1 - 5x_2 \geq 7$

Less than type constraints

Add a new positive variable (called a *slack* variable) to convert a $\leq$ constraint (LE) to an equality. For example $3x_1 + 5x_2 \leq 7$ is converted to $3x_1 + 5x_2 + x_3 = 7$ where $x_3 \geq 0$ is a slack variable

Greater than type constraints

Subtract a new positive variable (called a *surplus* variable) to convert a $\geq$ constraint (GE) to equality. For example $3x_1 + 5x_2 \geq 7$ is converted to $3x_1 + 5x_2 - x_3 = 7$ where $x_3 \geq 0$ is a surplus variable. Note that, since the right hand sides of the constraints is restricted to be positive, we cannot simply multiply both sides of the GE constraints to convert them into LE type as was done for the KT conditions.

Unrestricted variables

The standard LP form restricts all variables to be positive. If an actual optimization variable is unrestricted in sign it can be converted to the standard form by defining it as difference of two new positive variables. For example if variable $x_1$is unrestricted in sign it is replaced by two new variables $y_1$and $y_2$with $x_1 = y_1 - y_2$. Both the new variables are positive. After the solution is obtained, if $y_1 > y_2$then $x_1$will be positive and if $y_1 < y_2$then $x_1$will be negative.

Example

Convert the following problem to the standard LP form.

Maximize $z = 3x_1 + 8x_2$

Subject to $\begin{pmatrix} 3x_1 + 4x_2 \geq -20 \\ x_1 + 3x_2 \geq 6 \\ x_1 \geq 0 \end{pmatrix}$

Note that x_2 is unrestricted in sign.

Solution

Define new variables (all ≥ 0)

$x_1 = y_1 x_2 = y_2 - y_3$

Substituting these and multiplying the first constraint by a negative sign the problem is as follows.

Maximize $z = 3y_1 + 8y_2 - 8y_3$

Subject to $\begin{pmatrix} -3y_1 - 4y_2 + 4y_3 \leq 20 \\ y_1 + 3y_2 - 3y_3 \geq 6 \\ y_1,\ y_2,\ y_3 \geq 0 \end{pmatrix}$

Multiplying the objective function by a negative sign and introducing slack/surplus variables in the constraints, the problem in the standard LP form is as follows.

Minimize $f = -3y_1 - 8y_2 + 8y_3$

Subject to $\begin{pmatrix} -3y_1 - 4y_2 + 4y_3 + y_4 = 20 \\ y_1 + 3y_2 - 3y_3 - y_5 = 6 \\ y_1,\ \ldots,\ y_5 \geq 0 \end{pmatrix}$

Class activity

Convert the following problem to the standard LP form.

Maximize $z = x_1 + 4x_2$

Subject to $\begin{pmatrix} x_1 + 2x_2 \leq 5 \\ 2x_1 + x_2 = 4 \\ x_1 - x_2 \geq 3 \\ x_1 \geq 0 \\ x_2 \text{ free in sign} \end{pmatrix}$

Solution

Define new variables

$y_1 = x_1 \;\; y_2 - y_3 = x_2$

In terms of these new variables, the problem is as follows.

Maximize $z = y_1 + 4y_2 - 4y_3$

Subject to $\begin{pmatrix} y_1 + 2y_2 - 2y_3 \leq 5 \\ 2y_1 + y_2 - y_3 = 4 \\ y_1 - y_2 + y_3 \geq 3 \\ y_i \geq 0,\ i = 1, 2, 3 \end{pmatrix}$

Multiply objective function by negative sign to convert it to minimization form, add slack variable y_4 to the first constraints and subtract a surplus variable y_5 to the third constraint to convert them into equalities to get the following standard LP form.

Minimize $f = -y_1 - 4y_2 + 4y_3$

Subject to $\begin{pmatrix} y_1 + 2y_2 - 2y_3 + y_4 = 5 \\ 2y_1 + y_2 - y_3 = 4 \\ y_1 - y_2 + y_3 - y_5 = 3 \\ y_i \geq 0,\ i = 1, \ldots, 5 \end{pmatrix}$

Optimum of LP Problems

LP problem is a convex problem. (The procedure for checking convexity is covered in Chapter 3 but is not discussed in the lectures.) This means that if an optimum solution exists, it is a global optimum. Furthermore all we need is one solution. No need to try to find multiple solutions.

LP problems with the constraints in standard form represent a system of n equations in m unknowns.

If $m = n$ (i.e. the number of constraints is equal to number of optimization variables), then the solution for all variables is obtained from the solution of constraint equations and there is no consideration of the objective function. This situation clearly does not represent an optimization problem.

If $m > n$ does not make sense because in this case some of the constraints must be linearly dependent on the others.

Thus from an optimization point of view the only meaningful case is when the number of constraints is smaller than the number of variables (after the problem has been expressed in the standard LP form).

Basic Solutions of an LP Problem

LP Problem

- m = Number of constraints (equalities in standard form)
- n = Number of variables
- m < n - LP problem

From m equations at most we can solve for m variables in terms of the remaining $n - m$ variables. The variables that we choose to solve for are called *basic* and the remaining variables are called *nonbasic*.

Consider the following example.

Minimize $f = -x_1 + x_2$

Subject to $\begin{pmatrix} x_1 - 2x_2 \geq 2 \\ x_1 + x_2 \leq 4 \\ x_1 \leq 3 \\ x_i \geq 0,\ i = 1, 2 \end{pmatrix}$

In the standard LP form

Minimize $f = -x_1 + x_2$

Subject to $\begin{pmatrix} x_1 - 2x_2 - x_3 = 2 \\ x_1 + x_2 + x_4 = 4 \\ x_1 + x_5 = 3 \\ x_i \ge 0,\ i = 1,\ \ldots,\ 5 \end{pmatrix}$

where x_3 is a surplus variable for the first constraint and $x_4$and x_5 are slack variables for the two less than type constraints. The total number of variables, $n = 5$ and the number of equations, $m = 3$. Thus we can have 3 basic variables and 2 nonbasic variables. If we arbitrarily choose x_3, x_4 and x_5 as basic variables, a general solution of the constraint equations can readily be written as follows.

$$x_3 = -2 + x_1 - 2x_2 \quad x_4 = 4 - x_1 - x_2 \quad x_5 = 3 - x_1$$

The general solution is valid for any values of the nonbasic variables. Since all variables are positive and we are interested in minimizing objective function, we assign 0 values to nonbasic variables. A solution from the constraint equations obtained by setting nonbasic variables to zero is called a *basic solution*. Therefore one possible basic solution for the above example is as follows.

$$x_3 = -2 \quad x_4 = 4 \quad x_5 = 3$$

Since all variables must be ≥ 0, this basic solution is infeasible because x_3 is negative.

Let's find another basic solution by choosing (again arbitrarily) x_1, x_4 and $x_5$as basic variables and x_2 and $x_3$as nonbasic. By setting nonbasic variables to zero, we need to solve for the basic variables from the following equations.

$$x_1 = 2 \quad x_1 + x_4 = 4 \quad x_1 + x_5 = 3$$

It can easily be verified that the solution is $x_1 = 2$, $x_4 = 2$, and $x_5 = 1$. Since all variables have positive values, this basic solution is feasible as well.

The maximum number of possible basic solutions depends on the number of constraints and the number of variables in the problem and can be determined from the following equation.

Number of possible basic solutions = Binomial[n, m] $\equiv \frac{n!}{m!\,(n-m)!}$

where "!" stands for *factorial*. For the example problem where $m = 3$ and $n = 5$, therefore the maximum number of basic solutions is

$$\frac{5!}{3!\quad 2!} = \frac{5\times4\times3!}{3!\times2} = 10$$

All these basic solutions are computed from the constraint equations and are summarized in the following table. The set of basic variables for a particular solution is called a *basis* for that solution.

	Basis	Solution	Status	f
(1)	$\{x_1, x_2, x_3\}$	$\{3, 1, -1, 0, 0\}$	Infeasible	–
(2)	$\{x_1, x_2, x_4\}$	$\{3, \frac{1}{2}, 0, \frac{1}{2}, 0\}$	Feasible	$-\frac{5}{2}$
(3)	$\{x_1, x_2, x_5\}$	$\{\frac{10}{3}, \frac{2}{3}, 0, 0, -\frac{1}{3}\}$	Infeasible	–
(4)	$\{x_1, x_3, x_4\}$	$\{3, 0, 1, 1, 0\}$	Feasible	-3
(5)	$\{x_1, x_3, x_5\}$	$\{4, 0, 2, 0, -1\}$	Infeasible	–
(6)	$\{x_1, x_4, x_5\}$	$\{2, 0, 0, 2, 1\}$	Feasible	-2
(7)	$\{x_2, x_3, x_4\}$	$\{--\}$	NoSolution	–
(8)	$\{x_2, x_3, x_5\}$	$\{0, 4, -10, 0, 3\}$	Infeasible	–
(9)	$\{x_2, x_4, x_5\}$	$\{0, -1, 0, 5, 3\}$	Infeasible	–
(10)	$\{x_3, x_4, x_5\}$	$\{0, 0, -2, 4, 3\}$	Infeasible	–

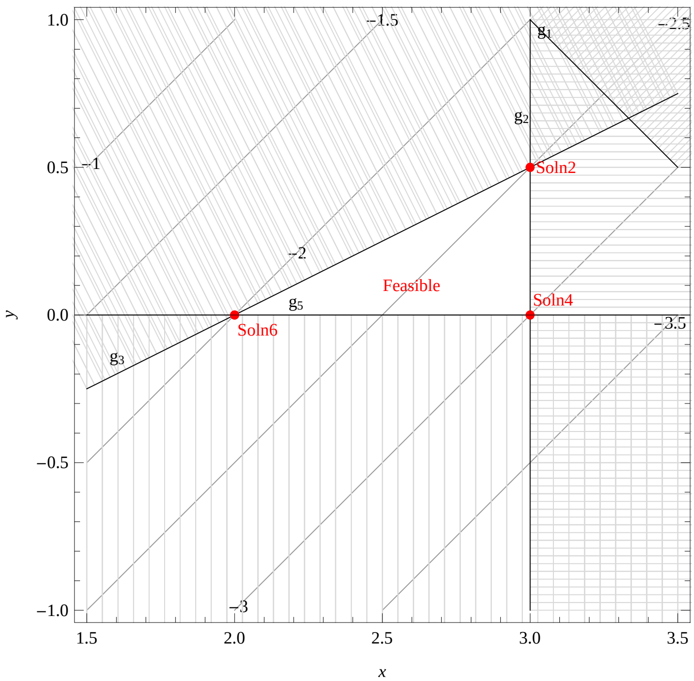
The 4th basic solution is feasible and has the lowest value of the objective function. Thus this represents the optimum solution for the problem.

Optimum solution:

$$x_1^* = 3 \quad x_2^* = 0 \quad x_3^* = 1 \quad x_4^* = 1 \quad x_5^* = 0 \quad f^* = -3$$

The third constraint is active because its slack variable x_5 is 0. Since x_3 and x_4 are positive, the first two constraints are inactive.

Since the original problem was a two variable problem, we can obtain a graphical solution to gain further insight into the basic feasible solutions. The three basic feasible solutions correspond to the three vertices of the feasible region. The infeasible basic solutions correspond to constraint intersections that are outside of the feasible region.



Class Activity

Compute all basic solutions of the following LP problem.

Minimize $\quad f = x_1 + x_2$

Subject to $\quad \begin{pmatrix} 2x_1 + x_2 \le 8 \\ 3x_1 + 2x_2 \le 10 \\ x_i \ge 0,\ i = 1,\ 2 \end{pmatrix}$

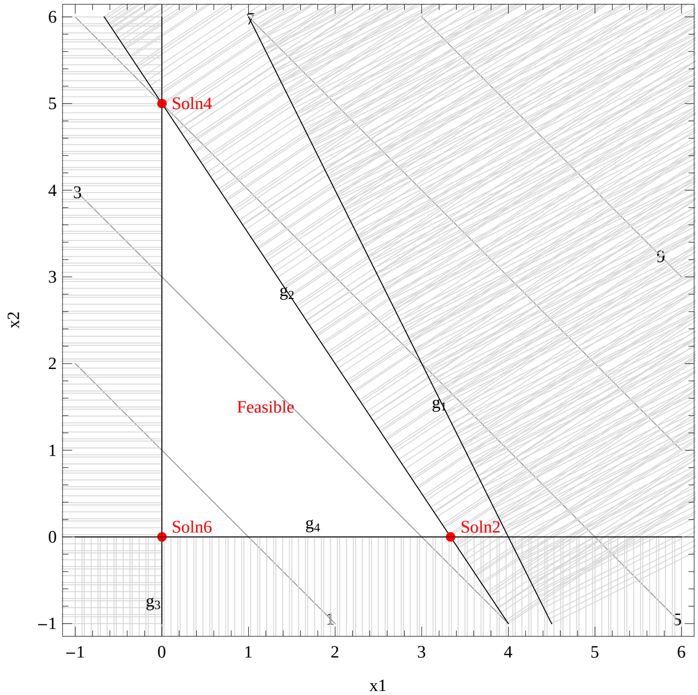
Solution

In the standard LP form

Minimize $\quad f = x_1 + x_2$

Subject to $\quad \begin{pmatrix} 2x_1 + x_2 + x_3 = 8 \\ 3x_1 + 2x_2 + x_4 = 10 \\ x_i \ge 0,\ i = 1,\ \ldots,\ 4 \end{pmatrix}$

Basis	Solution	Status	f
$\{x_1, x_2\}$	$\{6, -4, 0, 0\}$	Infeasible	–
$\{x_1, x_3\}$	$\{\frac{10}{3}, 0, \frac{4}{3}, 0\}$	Feasible	$\frac{10}{3}$
$\{x_1, x_4\}$	$\{4, 0, 0, -2\}$	Infeasible	–
$\{x_2, x_3\}$	$\{0, 5, 3, 0\}$	Feasible	5
$\{x_2, x_4\}$	$\{0, 8, 0, -6\}$	Infeasible	–
$\{x_3, x_4\}$	$\{0, 0, 8, 10\}$	Feasible	0



The Simplex Method

Basic Idea

Start with a basic feasible solution and then try to obtain neighboring basic feasible solution that has the objective function value lower than the current basic feasible solution.

At each try one of the current basic variables is made nonbasic and is replaced with a variable from the nonbasic set.

An optimum is reached when no other basic feasible solution can be found with lower objective function value.

Rules are established such that for most problems the method finds an optimum in a lot fewer steps than the total number of possible basic solutions.

The complete algorithm needs procedures for (a) finding a starting basic feasible solution, (b) bringing a currently nonbasic variable into the basic set, and (c) moving a currently basic variable out of the basic set to make room for the new basic variable.

Simplex Method for Problems with LE Constraints

Problems that initially have all less than type (LE, $\le$) constraints are easiest to deal with in the Simplex method. The procedure will be explained with reference to the following example.

Minimize $f = 5x_1 - 3x_2 - 8x_3$

Subject to
$$\begin{pmatrix} 2x_1 + 5x_2 - x_3 \le 1 \\ -2x_1 - 12x_2 + 3x_3 \le 9 \\ -3x_1 - 8x_2 + 2x_3 \le 4 \\ x_i \ge 0, i = 1, \ldots, 3 \end{pmatrix}$$

Starting basic feasible solution

The starting basic feasible solution is easy to obtain for problems that involve only LE type constraints (after making the right hand side positive, if necessary). A different slack variable must be added to each LE constraint to convert it into equality. We have as many slack variables as the number of constraint equations. If we treat these slack variables as basic variables and the actual variables as nonbasic (set to 0) then the right hand side of each constraint represents the basic solution. Since the right hand sides must all be positive (a requirement of the standard LP form) this basic solution is feasible as well.

For the example problem, introducing slack variables x_4, x_5, and x_6, the constraints are written in the standard LP form as follows.

$$\begin{aligned} 2x_1 + 5x_2 - x_3 + x_4 &= 1 \\ -2x_1 - 12x_2 + 3x_3 + x_5 &= 9 \\ -3x_1 - 8x_2 + 2x_3 + x_6 &= 4 \end{aligned}$$

Treating the slack variables as basic and the others as nonbasic, the starting basic feasible solution is

Basic: $x_4 = 1, x_5 = 9, x_6 = 4$ Nonbasic: $x_1 = x_2 = x_3 = 0$ $f = 0$

Note that the objective function is expressed in terms of nonbasic variables. In this first step we did not have to do anything special to achieve this. However, in subsequent steps, it is necessary to explicitly eliminate basic variables from the objective function.

Bringing a new variable into the basic set

In order to find a new basic feasible solution, one of the currently nonbasic variables must be made basic. The best candidate for this purpose is the variable that causes largest decrease in the objective function. The nonbasic variable that has the largest negative coefficient in the objective function is the best choice. This makes sense because the nonbasic variables are set to zero and so the current value of the objective function is not influenced by them. But if one of them is made basic, it will have a positive value and therefore if its coefficient is a large negative number it has the greatest potential of causing a large reduction in the objective function.

Continuing with the previous example we have the following situation.

Basic: $x_4 = 1, x_5 = 9, x_6 = 4$ Nonbasic: $x_1 = x_2 = x_3 = 0$

$f = 0 = 5x_1 - 3x_2 - 8x_3$

The largest negative coefficient is that of x_3. Thus our next basic feasible solution should use x_3 as one of its basic variables.

Moving an existing basic variable out of the basic set

The decision to remove a variable from the basic set is based on the need to keep the solution feasible. A little algebra shows that we should remove the basic variable that corresponds to the smallest ratio of the right hand side of constraints and the coefficients of the new variable that is being brought into the basic set. Furthermore if the coefficient is negative, then there is no danger that the associated basic variable will become negative, thus during this process we need to look at ratios of right hand sides and positive constraint coefficients.

To understand the reasoning behind this rule, let's continue with the example problem. The constraints and the corresponding basic variables, are as follows.

Constraint 1: $x_4$basic: $2x_1 + 5x_2 - x_3 + x_4 = 1$
Constraint 2: $x_5$basic: $-2x_1 - 12x_2 + 3x_3 + x_5 = 9$
Constraint 3: $x_6$basic: $-3x_1 - 8x_2 + 2x_3 + x_6 = 4$

The variable x_3 is to be brought into the basic set which means that it will have a value greater than or equal to zero in the next basic feasible solution. For constraint 1, the coefficient of $x_3$is negative and therefore the solution from this constraint will remain positive as a result of making $x_3$basic. In constraints 2 and 3 the coefficients of $x_3$are positive and therefore these constraints could result in negative values of variables. From the constraint 2, we see that the new basic variable $x_3$must have a value less than or equal to $9/3 = 3$ otherwise this constraint will give a negative solution. Similarly the third constraint shows that $x_3$must have a value less than or equal to $4/2 = 2$. The constraint 3 is more critical and therefore we should make x_3 a basic variable for this constraint and hence remove x_6 from the basic set.

The next basic feasible solution

Now we are in a position to compute the next basic feasible solution. We need to solve the system of constraint equations for the new set of basic variables. Also the objective function must be expressed in terms of new nonbasic variables in order to continue with the subsequent steps of the Simplex method.

For the example problem the constraints currently are written as follows.

Constraint 1: $x_4$basic: $2x_1 + 5x_2 - x_3 + x_4 = 1$
Constraint 2: $x_5$basic: $-2x_1 - 12x_2 + 3x_3 + x_5 = 9$
Constraint 3: $x_6$basic: $-3x_1 - 8x_2 + 2x_3 + x_6 = 4$

The new basic variable set is (x_4, x_5, x_3). We can achieve this by eliminating $x_3$from the first and the second constraints. We divide the third constraint by 2 first to make the coefficient of $x_3$equal to 1.

Constraint 3: $x_3$basic: $-\frac{3}{2}x_1 - 4x_2 + x_3 + \frac{1}{2}x_6 = 2$

This constraint is known as the *pivot row* for computing the new basic feasible solution and is used to eliminate $x_3$from the other constraints and the objective function. Variable $x_3$can be eliminated from constraint 1 by adding pivot row to the first constraint.

Constraint 1: $x_4$basic: $\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3$

From the second constraint variable x_3 is eliminated by adding (-3) times the pivot row to it.

Constraint 2: $x_5$basic: $\frac{5}{2}x_1 + x_5 - \frac{3}{2}x_6 = 3$

The objective function is

$$5x_1 - 3x_2 - 8x_3 = f$$

From this function variable x_3 is eliminated by adding 8 times the pivot row to it.

$$-7x_1 - 35x_2 + 4x_6 = f + 16$$

We now have a new basic feasible solution as follows.

Basic: $x_3 = 2$, $x_4 = 3$, $x_5 = 3$ Nonbasic: $x_1 = x_2 = x_6 = 0$ $f = -16$

Considering the objective function value this solution is better than our starting solution.

The optimum solution

The series of steps is repeated until all coefficients in the objective function become positive. When this happens then bringing any of the current nonbasic variables into the basic set will increase the objective function value. This indicates that we have reached the lowest value possible and the current basic feasible solution represents the optimum solution.

The next step for the example problem is summarized as follows.

Constraint 1: $x_4$basic: $\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3$

Constraint 2: $x_5$basic: $\frac{5}{2}x_1 + x_5 - \frac{3}{2}x_6 = 3$

Constraint 3: $x_3$basic: $-\frac{3}{2}x_1 - 4x_2 + x_3 + \frac{1}{2}x_6 = 2$

Objective: $-7x_1 - 35x_2 + 4x_6 = f + 16$

In the objective row, the variable $x_2$has the largest negative coefficient. Thus the next variable to be made basic is x_2. In the constraint expressions, $x_2$shows up only in the first constraint which has $x_4$as the basic variable. Thus $x_4$should be removed from the basic set. The coefficient of $x_2$in the first row is already 1 thus there is nothing that needs to be done to this equation. For eliminating $x_2$from the remaining equation now we must use the first equation as the pivot row (PR) as follows.

Constraint 1: $x_2$basic: $\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3 \quad (\text{PR})$

Constraint 2: $x_5$basic: $\frac{5}{2}x_1 + x_5 - \frac{3}{2}x_6 = 3 \quad (\text{no change})$

Constraint 3: $x_3$basic: $\frac{1}{2}x_1 + x_3 + 4x_4 + \frac{5}{2}x_6 = 14 \quad (\text{Added } 4\times\text{PR})$

Objective: $\frac{21}{2}x_1 + 35x_4 + \frac{43}{2}x_6 = f + 121 \quad (\text{Added } 35\times\text{PR})$

Looking at the objective function row we see that all coefficients are positive. This means that bringing any of the current nonbasic variables into the basic set will increase the objective function. Thus we have reached the lowest value possible and hence the above basic feasible solution represents the optimum solution.

Optimum solution: Basic: $x_2 = 3$, $x_3 = 14$, $x_5 = 3$ Nonbasic: $x_1 = x_4 = x_6 = 0$
$f + 121 = 0$ giving $f^* = -121$

It is interesting to note that the total number of possible basic solutions for this example was Binomial[6,3] = 20. However the Simplex method found the optimum in only 3 steps.

Simplex Tableau

A tabular form is convenient to organize the calculations involved in the simplex method. The rows represent coefficients of the constraint equations. The objective function is written in the last row. The first column indicates the basic variable associated with the constraint in that row. Recall that this variable should appear only in one constraint with a coefficient of 1. The last column represents the right hand sides of the equations. The other columns represent coefficients of variables, usually arranged in an ascending order with the actual variables first, followed by the slack variables. The exact form of the simplex tableau is illustrated through the following examples.

Example

Consider solution of the problem considered in the previous section using the tableau form.

$$\text{Minimize} \quad f = 5x_1 - 3x_2 - 8x_3$$

$$\text{Subject to} \quad \begin{pmatrix} 2x_1 + 5x_2 - x_3 \leq 1 \\ -2x_1 - 12x_2 + 3x_3 \leq 9 \\ -3x_1 - 8x_2 + 2x_3 \leq 4 \\ x_i \geq 0, i = 1, \ldots, 3 \end{pmatrix}$$

Introducing slack variables, the constraints are written in the standard LP form as follows.

$$\begin{aligned} &2x_1 + 5x_2 - x_3 + x_4 = 1 \\ &-2x_1 - 12x_2 + 3x_3 + x_5 = 9 \\ &-3x_1 - 8x_2 + 2x_3 + x_6 = 4 \end{aligned}$$

The starting tableau is as follows.

$$\text{Initial Tableau:} \begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_4 & 2 & 5 & -1 & 1 & 0 & 0 & 1 \\ x_5 & -2 & -12 & 3 & 0 & 1 & 0 & 9 \\ x_6 & -3 & -8 & 2 & 0 & 0 & 1 & 4 \\ \text{Obj.} & 5 & -3 & -8 & 0 & 0 & 0 & f \end{pmatrix}$$

The first three rows are simply the constraint equations. The fourth row is the objective function, expressed in the form of an equation, $5x_1 - 3x_2 - 8x_3 = f$. The right hand side of the objective function equation is set to f. Since x_4 appears only in the first row, it is the basic variable associated with the first constraint. Similarly the basic variables associated with the other two constraints are x_5 and x_6. The basic variables for each constraint row are identified in the first column.

From the tableau we can read the basic feasible solution simply by setting the basis to the rhs (since the nonbasic variables are all set to 0).

$$\text{Basic}: \quad x_4 = 1, x_5 = 9, x_6 = 4 \quad \text{Nonbasic}: x_1 = x_2 = x_3 = 0 \quad f = 0$$

We now proceed to the first iteration of the simplex method. To bring a new variable into the basic set, we look at the largest negative number in the objective function row. From the simplex tableau we can readily identify that the coefficient corresponding to x_3 is most negative (–8). Thus x_3 should be made basic.

The variable that must be removed from the basic set, corresponds to the smallest ratio of the entries in the constraint right hand sides and the positive entries in the column corresponding to the new variable to be made basic. From the simplex tableau, we see that in the column corresponding to x_3, there are two constraint rows that have positive coefficients. Ratios of the right hand side and these entries are as follows.

$$\text{Ratios: } \left\{ \tfrac{9}{3} = 3, \ \tfrac{4}{2} = 2 \right\}$$

The minimum ratio corresponds to the third constraint for which x_6 is the current basic variable. Thus we should make x_6 nonbasic.

Based on these decisions, our next tableau must be of the following form.

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_4 & - & - & 0 & 1 & 0 & - & - \\ x_5 & - & - & 0 & 0 & 1 & - & - \\ x_3 & - & - & 1 & 0 & 0 & - & - \\ \text{Obj.} & - & - & 0 & 0 & 0 & - & - \end{pmatrix}$$

That is we need to eliminate variable x_3 from the first, second, and the objective function row. Since each row represents an equation, this can be done by adding or subtracting appropriate multiples of rows together. However we must be careful in how we perform these steps because we need to preserve x_4 and x_5 as basic variables for the first and the second constraints. That is the form of columns x_4 and x_5 must be maintained during these row operations.

The systematic procedure to actually bring the tableau into the desired form is to first divide the row 3 (because it involves the new basic variable) by its diagonal element (coefficient corresponding to new basic variable). Thus dividing row 3 by 2 we have the following situation.

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_4 & - & - & 0 & 1 & 0 & - & - \\ x_5 & - & - & 0 & 0 & 1 & - & - \\ x_3 & -\frac{3}{2} & -4 & 1 & 0 & 0 & \frac{1}{2} & 2 \\ \text{Obj.} & 0 & - & - & - & 0 & 0 & - \end{pmatrix}$$

We call this modified row as *pivot row* (PR) and use it to eliminate x_3 from the other rows. The computations are as follows.

$$\begin{array}{cccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} & & \\ x_4 & \frac{1}{2} & 1 & 0 & 1 & 0 & \frac{1}{2} & 3 & \Leftarrow & \text{PR} + \text{Row1} \\ x_5 & \frac{5}{2} & 0 & 0 & 0 & 1 & -\frac{3}{2} & 3 & \Leftarrow & -3\times \text{PR} + \text{Row2} \\ x_3 & -\frac{3}{2} & -4 & 1 & 0 & 0 & \frac{1}{2} & 2 & \Leftarrow & \text{PR} \\ \text{Obj.} & -7 & -35 & 0 & 0 & 0 & 4 & 16+f & \Leftarrow & 8\times \text{PR} + \text{Obj. Row} \end{array}$$

This completes one step of the Simplex method and we have a second basic feasible solution.

$$\text{Second Tableau:} \begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_4 & \frac{1}{2} & 1 & 0 & 1 & 0 & \frac{1}{2} & 3 \\ x_5 & \frac{5}{2} & 0 & 0 & 0 & 1 & -\frac{3}{2} & 3 \\ x_3 & -\frac{3}{2} & -4 & 1 & 0 & 0 & \frac{1}{2} & 2 \\ \text{Obj.} & -7 & -35 & 0 & 0 & 0 & 4 & 16+f \end{pmatrix}$$

$$\text{Basic}: \; x_3 = 2, x_4 = 3, x_5 = 3 \quad \text{Nonbasic}: x_1 = x_2 = x_6 = 0 \quad f = -16$$

The same series of steps can now be repeated for additional tableaus. For the third tableau we should be make x_2 basic (largest negative coefficient in the obj. row = –35). In the column corresponding to x_2, only the first row has a positive coefficient and thus we have no choice but to make x_4 (current basic variable for first row) nonbasic. Based on these decisions, our next tableau must be of the following form.

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_2 & - & 1 & 0 & - & 0 & - & - \\ x_5 & - & 0 & 0 & - & 1 & - & - \\ x_3 & - & 0 & 1 & - & 0 & - & - \\ \text{Obj.} & - & 0 & 0 & - & 0 & - & - \end{pmatrix}$$

The first row already has a 1 in the x_2 column, therefore, we don't need to do anything and use it as our new pivot row to eliminate x_3 from the other rows. The computations are as follows.

$$\begin{array}{cccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} & & \\ x_2 & \frac{1}{2} & 1 & 0 & 1 & 0 & \frac{1}{2} & 3 & \Longleftarrow & \text{PR} \\ x_5 & \frac{5}{2} & 0 & 0 & 0 & 1 & -\frac{3}{2} & 3 & \Longleftarrow & \text{Row2} \\ x_3 & \frac{1}{2} & 0 & 1 & 4 & 0 & \frac{5}{2} & 14 & \Longleftarrow & 4\times\text{PR}+\text{Row3} \\ \text{Obj.} & \frac{21}{2} & 0 & 0 & 35 & 0 & \frac{43}{2} & 121+f & \Longleftarrow & 35\times\text{PR}+\text{Obj. Row} \end{array}$$

This completes the second iteration of the Simplex method and we have a third basic feasible solution.

$$\text{Third Tableau:} \begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_2 & \frac{1}{2} & 1 & 0 & 1 & 0 & \frac{1}{2} & 3 \\ x_5 & \frac{5}{2} & 0 & 0 & 0 & 1 & -\frac{3}{2} & 3 \\ x_3 & \frac{1}{2} & 0 & 1 & 4 & 0 & \frac{5}{2} & 14 \\ \text{Obj.} & \frac{21}{2} & 0 & 0 & 35 & 0 & \frac{43}{2} & 121+f \end{pmatrix}$$

$$\text{Basic}: \quad x_2 = 3,\ x_3 = 14,\ x_5 = 3 \quad \text{Nonbasic}: x_1 = x_4 = x_6 = 0 \quad f = -121$$

Since all coefficients in the Obj. row are positive, we cannot reduce the objective function any further and thus have reached the minimum. The optimum solution is

$$\text{Optimum}: \quad x_1 = 0,\ x_2 = 3,\ x_3 = 14,\ x_4 = 0,\ x_5 = 3,\ x_6 = 0 \quad f = -121$$

Class Activity

Solve the following LP problem using the tableau form of the Simplex method.

$$\text{Maximize} \qquad -7x_1 - 4x_2 + 15x_3$$

$$\text{Subject to} \quad \begin{pmatrix} \frac{x_1}{3} - \frac{32x_2}{9} + \frac{20x_3}{9} \le 1 \\ \frac{x_1}{6} - \frac{13x_2}{9} + \frac{5x_3}{18} \le 2 \\ \frac{2x_1}{3} - \frac{16x_2}{9} + \frac{x_3}{9} \ge -3 \\ x_i \ge 0,\ i = 1, \ldots, 3 \end{pmatrix}$$

Solution

Note that the problem as stated has a greater than type constraint. However since the right hand side of all constraints must be positive, as soon as we multiply the third constraint by a negative sign, all constraints become of LE type and therefore we can handle this problem with the procedure developed so far.

$$\frac{2x_1}{3} - \frac{16x_2}{9} + \frac{x_3}{9} \ge -3 \quad \text{same as} \quad -\frac{2x_1}{3} + \frac{16x_2}{9} - \frac{x_3}{9} \le 3$$

In the standard LP form

$$\text{Minimize} \quad f = 7x_1 + 4x_2 - 15x_3$$

$$\text{Subject to} \quad \begin{pmatrix} \frac{x_1}{3} - \frac{32x_2}{9} + \frac{20x_3}{9} + x_4 = 1 \\ \frac{x_1}{6} - \frac{13x_2}{9} + \frac{5x_3}{18} + x_5 = 2 \\ -\frac{2x_1}{3} + \frac{16x_2}{9} - \frac{x_3}{9} + x_6 = 3 \\ x_i \geq 0,\ i = 1, \ldots, 6 \end{pmatrix}$$

where x_4, x_5 and x_6 are slack variables for the three constraints.

$$\text{Initial Tableau:} \begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_4 & \frac{1}{3} & -\frac{32}{9} & \frac{20}{9} & 1 & 0 & 0 & 1 \\ x_5 & \frac{1}{6} & -\frac{13}{9} & \frac{5}{18} & 0 & 1 & 0 & 2 \\ x_6 & -\frac{2}{3} & \frac{16}{9} & -\frac{1}{9} & 0 & 0 & 1 & 3 \\ \text{Obj.} & 7 & 4 & -15 & 0 & 0 & 0 & f \end{pmatrix}$$

New basic variable = x_3(–15 is the largest negative number in the Obj. row)

Ratios: $\left\{\frac{1}{20/9} = 0.45,\ \frac{2}{5/18} = 7.2\right\}$ Minimum = 0.45 $\Longrightarrow x_4$ out of the basic set.

$$\begin{array}{cccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} & & \\ x_3 & \frac{3}{20} & -\frac{8}{5} & 1 & \frac{9}{20} & 0 & 0 & \frac{9}{20} & \Longleftarrow & \text{PR}\left(=\text{Row }1\big/\frac{20}{9}\right) \\ x_5 & \frac{1}{8} & -1 & 0 & -\frac{1}{8} & 1 & 0 & \frac{15}{8} & \Longleftarrow & -\frac{5}{18}\times\text{PR}+\text{Row2} \\ x_6 & -\frac{13}{20} & \frac{8}{5} & 0 & \frac{1}{20} & 0 & 1 & \frac{61}{20} & \Longleftarrow & \frac{1}{9}\times\text{PR}+\text{Row3} \\ \text{Obj.} & \frac{37}{4} & -20 & 0 & \frac{27}{4} & 0 & 0 & \frac{27}{4}+f & \Longleftarrow & 15\times\text{PR}+\text{Obj. Row} \end{array}$$

This completes one step of the Simplex method and we have a second basic feasible solution.

$$\text{Second Tableau:} \begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_3 & \frac{3}{20} & -\frac{8}{5} & 1 & \frac{9}{20} & 0 & 0 & \frac{9}{20} \\ x_5 & \frac{1}{8} & -1 & 0 & -\frac{1}{8} & 1 & 0 & \frac{15}{8} \\ x_6 & -\frac{13}{20} & \frac{8}{5} & 0 & \frac{1}{20} & 0 & 1 & \frac{61}{20} \\ \text{Obj.} & \frac{37}{4} & -20 & 0 & \frac{27}{4} & 0 & 0 & \frac{27}{4}+f \end{pmatrix}$$

New basic variable = x_2(–20 is the largest negative number in the Obj. row)

Ratios: $\left\{\frac{61/20}{8/5}\right\}$ Minimum (only choice) $\Longrightarrow x_6$ out of the basic set.

$$\begin{array}{cccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} & & \\ x_3 & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{7}{2} & \Longleftarrow & \frac{8}{5}\times\text{PR}+\text{Row1} \\ x_5 & -\frac{9}{32} & 0 & 0 & -\frac{3}{32} & 1 & \frac{5}{8} & \frac{121}{32} & \Longleftarrow & \text{PR}+\text{Row2} \\ x_2 & -\frac{13}{32} & 1 & 0 & \frac{1}{32} & 0 & \frac{5}{8} & \frac{61}{32} & \Longleftarrow & \text{PR}\left(=\text{Row3}\big/\frac{8}{5}\right) \\ \text{Obj.} & \frac{9}{8} & 0 & 0 & \frac{59}{8} & 0 & \frac{25}{2} & \frac{359}{8}+f & \Longleftarrow & 20\times\text{PR}+\text{Obj. Row} \end{array}$$

Third Tableau:
$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_3 & -\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{7}{2} \\ x_5 & -\frac{9}{32} & 0 & 0 & -\frac{3}{32} & 1 & \frac{5}{8} & \frac{121}{32} \\ x_2 & -\frac{13}{32} & 1 & 0 & \frac{1}{32} & 0 & \frac{5}{8} & \frac{61}{32} \\ \text{Obj.} & \frac{9}{8} & 0 & 0 & \frac{59}{8} & 0 & \frac{25}{2} & \frac{359}{8}+f \end{pmatrix}$$

Since all coefficients in the Obj. row are positive, we cannot reduce the objective function any further and we have reached the minimum. The optimum solution is

$$\text{Optimum}: \quad x_1 = 0,\ x_2 = \frac{61}{32},\ x_3 = \frac{7}{2},\ x_4 = 0,\ x_5 = \frac{121}{32},\ x_6 = 0 \quad f = -\frac{359}{8}$$

Simplex Method for Problems with GE or EQ Constraints

The starting basic feasible solution is more difficult to obtain for problems that involve greater than (GE) type (after making the right hand side positive, if necessary) or equality (EQ) constraints. The reason is that there is no unique positive variable associated with each constraint. A unique surplus variable is present in each GE constraint but it is multiplied by a negative sign and thus will give an infeasible solution if treated as basic variable. An equality constraint does not need a slack/surplus variable and thus one cannot assume that there will always be a unique variable for each equality constraint.

The situation is handled by what is known as the *Phase I* simplex method. A unique artificial variable is added to each GE and EQ type constraint. Treating these artificial variables as basic and the actual variables as nonbasic gives a starting basic feasible solution. An artificial objective function, denoted by $\phi(\mathbf{x})$, is defined as the sum of all artificial variables needed in the problem. During this so-called phase I, this artificial objective function is minimized using the usual simplex procedure. Since there are no real constraints on ϕ, the optimum solution of phase I is reached when $\phi = 0$. That is when all artificial variables are equal to zero (out of the basis) which is the lowest value possible because all variables are positive in LP. This optimum solution of phase I is a basic feasible solution for the original problem since when the artificial variables are set to zero, the original constraints are recovered. Using this basic feasible solution we are then in a position to start solution of the original problem with the actual objective function. This is known as *Phase II* and is same as that described for LE constraints in the previous section.

Example

Consider the following example with two GE constraints.

$$\text{Minimize} \qquad f = 2x_1 + 4x_2 + 3x_3$$

$$\text{Subject to} \qquad \begin{pmatrix} -x_1 + x_2 + x_3 \geq 2 \\ 2x_1 + x_2 \geq 1 \\ x_i \geq 0,\ i = 1, \ldots, 3 \end{pmatrix}$$

Introducing surplus variables x_4 and x_5, the constraints are written in the standard LP form as follows.

$$-x_1 + x_2 + x_3 - x_4 = 2 \quad 2x_1 + x_2 - x_5 = 1$$

Now introducing artificial variables x_6 and x_7, the Phase I objective function and constraints are as follows.

Phase I Problem:

Minimize $\phi = x_6 + x_7$

Subject to $\begin{pmatrix} -x_1 + x_2 + x_3 - x_4 + x_6 = 2 \\ 2x_1 + x_2 - x_5 + x_7 = 1 \\ x_i \geq 0,\ i = 1,\ \ldots,\ 7 \end{pmatrix}$

The starting basic feasible solution for Phase I obviously is as follows.

Basic: $x_6 = 2,\ x_7 = 1$ Nonbasic: $x_1 = x_2 = x_3 = x_4 = x_5 = 0 \quad \phi = 3$

Before proceeding with the simplex method, the artificial objective function must be expressed in terms of nonbasic variables. It can easily be done by solving for the artificial variables from the constraint equations and substituting into the artificial objective function.

From the constraints we have

$$x_6 = 2 + x_1 - x_2 - x_3 + x_4 \qquad x_7 = 1 - 2x_1 - x_2 + x_5$$

Thus the artificial objective function is written as

$$\phi = x_6 + x_7 = 3 - x_1 - 2x_2 - x_3 + x_4 + x_5$$

or $\phi - 3 = -x_1 - 2x_2 - x_3 + x_4 + x_5$

Obviously the actual objective function is not needed during the Phase I. However, all reduction operations are performed on it as well so that at the end of Phase I, f is in the correct form (that is it expressed in terms on nonbasic variables only) for the Simplex method. The complete solution is as follows.

Phase I: Initial Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_6 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ x_7 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 2 & 4 & 3 & 0 & 0 & 0 & 0 & f \\ \text{ArtObj.} & -1 & -2 & -1 & 1 & 1 & 0 & 0 & -3+\phi \end{pmatrix}$$

New basic variable = x_2(–2 is the largest negative number in the ArtObj. row)

Ratios: $\left\{\frac{2}{1} = 2,\ \frac{1}{1} = 1\right\}$ Minimum = 1 $\Longrightarrow x_7$ out of the basic set.

Phase I: Second Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_6 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & -6 & 0 & 3 & 0 & 4 & 0 & -4 & -4+f \\ \text{ArtObj.} & 3 & 0 & -1 & 1 & -1 & 0 & 2 & -1+\phi \end{pmatrix} \begin{array}{lc} \Leftarrow & -\text{PR} + \text{Row1} \\ \Leftarrow & \text{PR} \\ \Leftarrow & -4\times \text{PR} + \text{Obj. Row} \\ \Leftarrow & 2\times \text{PR} + \text{ArtObj. Row} \end{array}$$

New basic variable = x_3(–1 is the first largest negative number in the ArtObj. row)

Ratios: $\left\{\frac{1}{1}\right\}$ Minimum (only choice) $\Longrightarrow x_6$ out of the basic set.

Phase I: Third Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_3 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 3 & 0 & 0 & 3 & 1 & -3 & -1 & -7+f \\ \text{ArtObj.} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \phi \end{pmatrix} \begin{array}{l} \\ \Leftarrow \quad \text{PR} \\ \Leftarrow \quad \text{Row2} \\ \Leftarrow \quad -3\times \text{PR} + \text{Obj. Row} \\ \Leftarrow \quad \text{PR} + \text{ArtObj. Row} \end{array}$$

All coefficient in the artificial objective function row are now positive signalling that the optimum of Phase I has reached. The solution is as follows.

Basic: $x_2 = 1 \quad x_3 = 1$ Nonbasic: $x_1 = x_4 = \ldots = x_7 = 0 \, \phi = 0$

Since the artificial variables are now zero, the constraint equations now represent the original constraints and we have a basic feasible solution for our original problem.

Phase II with the actual objective function can now begin. Ignoring the artificial objective function row, we have the following initial simplex tableau for phase II. Note that the columns associated with artificial variables are really not needed anymore either. However, as will be seen later, the entries in these columns are useful in the sensitivity analysis. Thus we carry these columns through phase II as well. However we don't use these columns for any decision making.

Phase II: Initial Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_3 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 3 & 0 & 0 & 3 & 1 & -3 & -1 & -7+f \end{pmatrix}$$

All coefficient in the objective function row (excluding the artificial variables) are positive meaning that we cannot find another basic feasible solution without increasing objective function value. Thus this basic feasible solution is the optimum solution of the problem and we are done.

Optimum solution:

$$x_1 = 0,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0,\ \ x_5 = 0 \quad f = 7$$

Class Activity

Find the maximum of the following LP problem using simplex method. Note that variable x2 is not restricted to be positive.

Maximize $-3x_1 + 2x_2 - 4x_3 + x_4 - x_5$

Subject to

$$2x_1 + 3x_2 + x_3 + 4x_4 + 4x_5 = 12$$

$$4x_1 - 5x_2 + 3x_3 - x_4 - 4x_5 = 10$$

$$3x_1 - x_2 + 2x_3 + 2x_4 + x_5 \geq 8$$

All variables must be ≥ 0 except for x_2 which is unrestricted in sign.

Solution

Minimize $3x_1 - 2x_2 + 2x_3 + 4x_4 - x_5 + x_6$

$$\text{Subject to}\begin{pmatrix} 2x_1 + 3x_2 - 3x_3 + x_4 + 4x_5 + 4x_6 = 12 \\ 4x_1 - 5x_2 + 5x_3 + 3x_4 - x_5 - 4x_6 = 10 \\ 3x_1 - x_2 + x_3 + 2x_4 + 2x_5 + x_6 \ge 8 \end{pmatrix}$$

All variables ≥ 0

∗∗∗∗∗∗∗∗∗∗ Initial simplex tableau ∗∗∗∗∗∗∗∗∗∗

The third constraint needs a surplus variable. This is placed in the 7th column of the tableau. All three constraints need artificial variables to start the phase I solution. These are placed in the last three columns. The artificial objective function is the sum of the three artificial variables. As explained earlier it is then expressed in terms of nonbasic variables to give the form included in the following tableau.

New problem variables: $\{x_1, x_2, x_3, x_4, x_5, x_6, s_3, a_1, a_2, a_3\}$

Basis	1	2	3	4	5	6	7	8	9	10	RHS
8	2	3	−3	1	4	4	0	1	0	0	12
9	4	−5	5	3	−1	−4	0	0	1	0	10
10	3	−1	1	2	2	1	−1	0	0	1	8
Obj.	3	−2	2	4	−1	1	0	0	0	0	f
ArtObj.	−9	3	−3	−6	−5	−1	1	0	0	0	$\phi - 30$

Variable to be made basic → 1

Ratios: RHS/Column 1 → $\begin{pmatrix} 6 & \frac{5}{2} & \frac{8}{3} \end{pmatrix}$

Variable out of the basic set → 9

∗∗∗∗∗∗∗∗∗∗Phase I − Iteration 1∗∗∗∗∗∗∗∗∗∗

Basis	1	2	3	4	5	6	7	8	9	10	RHS
8	0	$\frac{11}{2}$	$-\frac{11}{2}$	$-\frac{1}{2}$	$\frac{9}{2}$	6	0	1	$-\frac{1}{2}$	0	7
1	1	$-\frac{5}{4}$	$\frac{5}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	−1	0	0	$\frac{1}{4}$	0	$\frac{5}{2}$
10	0	$\frac{11}{4}$	$-\frac{11}{4}$	$-\frac{1}{4}$	$\frac{11}{4}$	4	−1	0	$-\frac{3}{4}$	1	$\frac{1}{2}$
Obj.	0	$\frac{7}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$	$-\frac{1}{4}$	4	0	0	$-\frac{3}{4}$	0	$f - \frac{15}{2}$
ArtObj.	0	$-\frac{33}{4}$	$\frac{33}{4}$	$\frac{3}{4}$	$-\frac{29}{4}$	−10	1	0	$\frac{9}{4}$	0	$\phi - \frac{15}{2}$

Variable to be made basic → 6

Ratios: RHS/Column 6 → $\begin{pmatrix} \frac{7}{6} & \infty & \frac{1}{8} \end{pmatrix}$

Variable out of the basic set → 10

∗∗∗∗∗∗∗∗∗∗Phase I − Iteration 2∗∗∗∗∗∗∗∗∗∗

Basis	1	2	3	4	5	6	7	8	9	10	RHS
8	0	$\frac{11}{8}$	$-\frac{11}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	0	$\frac{3}{2}$	1	$\frac{5}{8}$	$-\frac{3}{2}$	$\frac{25}{4}$
1	1	$-\frac{9}{16}$	$\frac{9}{16}$	$\frac{11}{16}$	$\frac{7}{16}$	0	$-\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{21}{8}$
6	0	$\frac{11}{16}$	$-\frac{11}{16}$	$-\frac{1}{16}$	$\frac{11}{16}$	1	$-\frac{1}{4}$	0	$-\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{8}$
Obj.	0	−1	1	2	−3	0	1	0	0	−1	$f - 8$
ArtObj.	0	$-\frac{11}{8}$	$\frac{11}{8}$	$\frac{1}{8}$	$-\frac{3}{8}$	0	$-\frac{3}{2}$	0	$\frac{3}{8}$	$\frac{5}{2}$	$\phi - \frac{25}{4}$

Variable to be made basic → 7

Ratios: RHS/Column 7 $\to \begin{pmatrix} \frac{25}{6} & \infty & \infty \end{pmatrix}$

Variable out of the basic set $\to$ 8

**********Phase I – Iteration 3**********

$$\begin{pmatrix} \text{Basis} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \text{RHS} \\ - & - & - & - & - & - & - & - & - & - & - & - \\ 7 & 0 & \frac{11}{12} & -\frac{11}{12} & -\frac{1}{12} & \frac{1}{4} & 0 & 1 & \frac{2}{3} & \frac{5}{12} & -1 & \frac{25}{6} \\ 1 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{11}{3} \\ 6 & 0 & \frac{11}{12} & -\frac{11}{12} & -\frac{1}{12} & \frac{3}{4} & 1 & 0 & \frac{1}{6} & -\frac{1}{12} & 0 & \frac{7}{6} \\ \text{Obj.} & 0 & -\frac{23}{12} & \frac{23}{12} & \frac{25}{12} & -\frac{13}{4} & 0 & 0 & -\frac{2}{3} & -\frac{5}{12} & 0 & \text{f} - \frac{73}{6} \\ \text{ArtObj.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \phi \end{pmatrix}$$

End of phase I

Variable to be made basic $\to$ 5

Ratios: RHS/Column 5 $\to \begin{pmatrix} \frac{50}{3} & \frac{22}{3} & \frac{14}{9} \end{pmatrix}$

Variable out of the basic set $\to$ 6

**********Phase II – Iteration 1**********

$$\begin{pmatrix} \text{Basis} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \text{RHS} \\ - & - & - & - & - & - & - & - & - & - & - & - \\ 7 & 0 & \frac{11}{18} & -\frac{11}{18} & -\frac{1}{18} & 0 & -\frac{1}{3} & 1 & \frac{11}{18} & \frac{4}{9} & -1 & \frac{34}{9} \\ 1 & 1 & -\frac{17}{18} & \frac{17}{18} & \frac{13}{18} & 0 & -\frac{2}{3} & 0 & \frac{1}{18} & \frac{2}{9} & 0 & \frac{26}{9} \\ 5 & 0 & \frac{11}{9} & -\frac{11}{9} & -\frac{1}{9} & 1 & \frac{4}{3} & 0 & \frac{2}{9} & -\frac{1}{9} & 0 & \frac{14}{9} \\ \text{Obj.} & 0 & \frac{37}{18} & -\frac{37}{18} & \frac{31}{18} & 0 & \frac{13}{3} & 0 & \frac{1}{18} & -\frac{7}{9} & 0 & \text{f} - \frac{64}{9} \end{pmatrix}$$

Variable to be made basic $\to$ 3

Ratios: RHS/Column 3 $\to \begin{pmatrix} \infty & \frac{52}{17} & \infty \end{pmatrix}$

Variable out of the basic set $\to$ 1

**********Phase II – Iteration 2**********

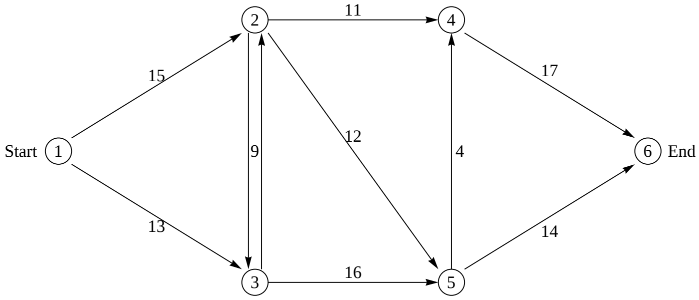
$$\begin{pmatrix} \text{Basis} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \text{RHS} \\ - & - & - & - & - & - & - & - & - & - & - & - \\ 7 & \frac{11}{17} & 0 & 0 & \frac{7}{17} & 0 & -\frac{13}{17} & 1 & \frac{11}{17} & \frac{10}{17} & -1 & \frac{96}{17} \\ 3 & \frac{18}{17} & -1 & 1 & \frac{13}{17} & 0 & -\frac{12}{17} & 0 & \frac{1}{17} & \frac{4}{17} & 0 & \frac{52}{17} \\ 5 & \frac{22}{17} & 0 & 0 & \frac{14}{17} & 1 & \frac{8}{17} & 0 & \frac{5}{17} & \frac{3}{17} & 0 & \frac{90}{17} \\ \text{Obj.} & \frac{37}{17} & 0 & 0 & \frac{56}{17} & 0 & \frac{49}{17} & 0 & \frac{3}{17} & -\frac{5}{17} & 0 & \text{f} - \frac{14}{17} \end{pmatrix}$$

Optimum solution $\to \left(\left\{ x1 \to 0,\ x2 \to -\frac{52}{17},\ x3 \to 0,\ x4 \to \frac{90}{17},\ x5 \to 0 \right\} \right)$

Optimum objective function value $\to -\frac{14}{17}$

Shortest route problem

This example demonstrates formulation and solution of an important class of problems known as *network* problems in the LP literature. In these problems a network of *nodes* and *links* is given. The problem is usually to find the maximum flow or the shortest route. As an example consider the problem of finding the shortest route between two cities while traveling on a given network of available roads. A typical situation is shown in Figure 0.0. The nodes represent cities and the links are the roads that connect these cities. The distances in kilo-meters along each road are noted in the figure.



Network diagram showing distances and direction of travel between cities

The optimization variables are the roads that one can take to reach the destination. Indicating the roads by the indices of the nodes that they connect, with the order indicating the direction of travel, we have the following set of optimization variables. Note that two separate variables are needed for the roads where travel in either direction is possible.

$$\text{Variables} = \{x_{12},\ x_{13},\ x_{23},\ x_{32},\ x_{24},\ x_{25},\ x_{35},\ x_{54}, x_{46}, x_{56}\}$$

The objective function is to minimize the distance travelled and is simply the sum of miles along each route as follows.

$$\text{Minimize } f = 15x_{12} + 13x_{13} + 9x_{23} + 9x_{32} + 11x_{24} + 12x_{25} + 16x_{35} + 4x_{54} + 17x_{46} + 14x_{56}$$

The constraints express the relationship between the links. The *inflow* at a node must equal the *outflow*.

$$\begin{aligned}
\text{Node 2}: &\quad x_{12} + x_{32} = x_{24} + x_{25} + x_{23} \\
\text{Node 3}: &\quad x_{13} + x_{23} = x_{32} + x_{35} \\
\text{Node 4}: &\quad x_{24} + x_{54} = x_{46} \\
\text{Node 5}: &\quad x_{35} + x_{25} = x_{54} + x_{56}
\end{aligned}$$

The origin and destination nodes are indicated by the fact that the outflow from the origin node and inflow into the destination node are equal to 1.

$$\begin{aligned}
\text{Node 1}: &\quad x_{12} + x_{13} = 1 \\
\text{Node 6}: &\quad x_{46} + x_{56} = 1
\end{aligned}$$

Solution

Problem variables redefined as: $\{x_{12} \to x_1,\ x_{13} \to x_2,\ x_{23} \to x_3,\ x_{32} \to x_4,\ x_{24} \to x_5,\ x_{25} \to x_6,\ x_{35} \to x_7,\ x_{54} \to x_8,\ x_{46} \to x_9,\ x_{56} \to x_{10}\}$

Minimize $15x_1 + 14x_{10} + 13x_2 + 9x_3 + 9x_4 + 11x_5 + 12x_6 + 16x_7 + 4x_8 + 17x_9$

$$\text{Subject to } \begin{pmatrix} x_1 - x_3 + x_4 - x_5 - x_6 = 0 \\ x_2 + x_3 - x_4 - x_7 = 0 \\ x_5 + x_8 - x_9 = 0 \\ -x_{10} + x_6 + x_7 - x_8 = 0 \\ x_1 + x_2 = 1 \\ x_{10} + x_9 = 1 \end{pmatrix}$$

All variables ≥ 0

********** Initial simplex tableau **********

New problem variables:{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, a_1, a_2, a_3, a_4, a_5, a_6}

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
11	1	0	−1	1	−1	−1	0	0	0	0	1	0	0	0	0	0	0
12	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
13	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
14	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
16	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	15	13	9	9	11	12	16	4	17	14	0	0	0	0	0	0	f
ArtObj.	−2	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\phi - 2$

Variable to be made basic → 1

Ratios: RHS/Column 1 → (0 ∞ ∞ ∞ 1 ∞)

Variable out of the basic set → 11

**********Phase I − Iteration 1**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
1	1	0	−1	1	−1	−1	0	0	0	0	1	0	0	0	0	0	0
12	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
13	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
14	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	0	1	1	−1	1	1	0	0	0	0	−1	0	0	0	1	0	1
16	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	13	24	−6	26	27	16	4	17	14	−15	0	0	0	0	0	f
ArtObj.	0	−2	−2	2	−2	−2	0	0	0	0	2	0	0	0	0	0	$\phi - 2$

Variable to be made basic → 2

Ratios: RHS/Column 2 → (∞ 0 ∞ ∞ 1 ∞)

Variable out of the basic set → 12

**********Phase I − Iteration 2**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
1	1	0	−1	1	−1	−1	0	0	0	0	1	0	0	0	0	0	0
2	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
13	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
14	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	0	0	0	0	1	1	1	0	0	0	−1	−1	0	0	1	0	1
16	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	0	11	7	26	27	29	4	17	14	−15	−13	0	0	0	0	f
ArtObj.	0	0	0	0	−2	−2	−2	0	0	0	2	2	0	0	0	0	$\phi - 2$

Variable to be made basic → 5

Ratios: RHS/Column 5 → (∞ ∞ 0 ∞ 1 ∞)

Variable out of the basic set → 13

**********Phase I − Iteration 3**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
1	1	0	−1	1	0	−1	0	1	−1	0	1	0	1	0	0	0	0
2	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
14	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	0	0	0	0	0	1	1	−1	1	0	−1	−1	−1	0	1	0	1
16	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	0	11	7	0	27	29	−22	43	14	−15	−13	−26	0	0	0	f
ArtObj.	0	0	0	0	0	−2	−2	2	−2	0	2	2	2	0	0	0	$\phi - 2$

Variable to be made basic → 6

Ratios: RHS/Column 6 → (∞ ∞ ∞ 0 1 ∞)

Variable out of the basic set → 14

**********Phase I − Iteration 4**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
1	1	0	−1	1	0	0	1	0	−1	−1	1	0	1	1	0	0	0
2	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	0	0	0	0	0	0	0	0	1	1	−1	−1	−1	−1	1	0	1
16	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	0	11	7	0	0	2	5	43	41	−15	−13	−26	−27	0	0	f
ArtObj.	0	0	0	0	0	0	0	0	−2	−2	2	2	2	2	0	0	$\phi - 2$

Variable to be made basic → 9

Ratios: RHS/Column 9 → (∞ ∞ ∞ ∞ 1 1)

Variable out of the basic set → 16

**********Phase I − Iteration 5**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
1	1	0	−1	1	0	0	1	0	0	0	1	0	1	1	0	1	1
2	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	1	0	0	1	0	1	0	0	1	0	0	1	1
6	0	0	0	0	0	1	1	−1	0	−1	0	0	0	1	0	0	0
15	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1	1	−1	0
9	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	0	11	7	0	0	2	5	0	−2	−15	−13	−26	−27	0	−43	f − 43
ArtObj.	0	0	0	0	0	0	0	0	0	0	2	2	2	2	0	2	ϕ

Note that the variable 15, which is the artificial variable for the 5th constraint, is still in the basis. However since it has a zero rhs, the artificial objective function value is reduced to 0 and we are done with the phase I. Furthermore we also notice that in the same constraint row (5th) all coefficients corresponding to non-artificial variables (columns 1 through 10) are 0. This indicates that this constraint is redundant and can be removed from the subsequent iterations. For ease of implementation, however, this constraint is kept in the following tableaus.

End of phase I

Variable to be made basic → 10

Ratios: RHS/Column 10 → (∞ ∞ 1 ∞ ∞ 1)

Variable out of the basic set → 9

**********Phase II − Iteration 1**********

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	RHS
−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−
1	1	0	−1	1	0	0	1	0	0	0	1	0	1	1	0	1	1
2	0	1	1	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0
5	0	0	0	0	1	0	0	1	−1	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	1	−1	1	0	0	0	0	1	0	1	1
15	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1	1	−1	0
10	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1
Obj.	0	0	11	7	0	0	2	5	2	0	−15	−13	−26	−27	0	−41	f − 41

Optimum solution → ($\{x_{12} \to 1,\ x_{13} \to 0,\ x_{23} \to 0,\ x_{32} \to 0,\ x_{24} \to 0,\ x_{25} \to 1,\ x_{35} \to 0,\ x_{54} \to 0,\ x_{46} \to 0,\ x_{56} \to 1\}$)

Optimum objective function value → 41

The solution indicates that the shortest route is 1 → 2, 2 → 5, and 5 → 6 with a total distance of 41 kilometers.

Extra