

MATH 337 SPRING 2013
HOMEWORK #12
DUE: 25 APRIL 2013

1. PRACTICE QUESTIONS

- 8.3 # 1-21 (odd)
- 8.4 # 1-11, 17-39 (odd)

2. ASSIGNMENT QUESTIONS

- 8.3 # 12 (5.3.12), 16 (5.3.16) ,
- 8.4 # 20 (5.4.4), 36 (5.5.18), 44 (5.4.22)

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x . Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x . Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x .

PROBLEMS

In each of Problems 1 through 4 determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{iv}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial value problem.

- $y'' + xy' + y = 0$; $y(0) = 1$, $y'(0) = 0$
- $y'' + (\sin x)y' + (\cos x)y = 0$; $y(0) = 0$, $y'(0) = 1$
- $x^2 y'' + (1+x)y' + 3(\ln x)y = 0$; $y(1) = 2$, $y'(1) = 0$
- $y'' + x^2 y' + (\sin x)y = 0$; $y(0) = a_0$, $y'(0) = a_1$

In each of Problems 5 through 8 determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

- $y'' + 4y' + 6xy = 0$; $x_0 = 0$, $x_0 = 4$
- $(x^2 - 2x - 3)y'' + xy' + 4y = 0$; $x_0 = 4$, $x_0 = -4$, $x_0 = 0$
- $(1 + x^3)y'' + 4xy' + y = 0$; $x_0 = 0$, $x_0 = 2$
- $xy'' + y = 0$; $x_0 = 1$
- Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 14 of Section 5.2.
- The Chebyshev Equation.** The Chebyshev⁷ differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

- Determine two linearly independent solutions in powers of x for $|x| < 1$.
- Show that if α is a nonnegative integer n , then there is a polynomial solution of degree n . These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems requiring a polynomial approximation to a function defined on $-1 \leq x \leq 1$.
- Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2$, and 3 .

For each of the differential equations in Problems 11 through 14 find the first four nonzero terms in each of two linearly independent power series solutions about the origin. What do you expect the radius of convergence to be for each solution?

- $y'' + (\sin x)y = 0$
- $e^x y'' + xy = 0$
- $(\cos x)y'' + xy' - 2y = 0$
- $e^{-x} y'' + \ln(1+x)y' - xy = 0$
- Suppose that you are told that x and x^2 are solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point?
Hint: Use Theorem 3.2.1, and note the values of x and x^2 at $x = 0$.

⁷Pafnuty L. Chebyshev (1821–1894), professor at Petersburg University for 35 years and the most influential nineteenth century Russian mathematician, founded the so-called “Petersburg school,” which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

First Order Equations. The series methods discussed in this section are directly applicable to the first order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0 , if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P . In each of Problems 16 through 21 solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problems 20 and 21 involve nonhomogeneous differential equations to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

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|--|-----------------------|
| 16. $y' - y = 0$ | 17. $y' - xy = 0$ |
| 18. $y' = e^{x^2}y$, three terms only | 19. $(1-x)y' = y$ |
| 20. $y' - y = x^2$ | 21. $y' + xy = 1 + x$ |

The Legendre Equation. Problems 22 through 29 deal with the Legendre⁸ equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 3, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that it is necessary to consider only $\alpha > -1$ because if $\alpha \leq -1$, then the substitution $\alpha = -(1 + \gamma)$ where $\gamma \geq 0$ leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

22. Show that two linearly independent solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\alpha(\alpha - 2)(\alpha - 4) \cdots (\alpha - 2m + 2)(\alpha + 1)(\alpha + 3) \cdots (\alpha + 2m - 1)}{(2m)!} x^{2m},$$

$$y_2(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha - 1)(\alpha - 3) \cdots (\alpha - 2m + 1)(\alpha + 2)(\alpha + 4) \cdots (\alpha + 2m)}{(2m + 1)!} x^{2m+1}.$$

23. Show that, if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x . Find the polynomials corresponding to $\alpha = 0, 2$, and 4 . Show that, if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x . Find the polynomials corresponding to $\alpha = 1, 3$, and 5 .
- ▶ 24. The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.
- Using the results of Problem 23, find the Legendre polynomials $P_0(x), \dots, P_5(x)$.
 - Plot the graphs of $P_0(x), \dots, P_5(x)$ for $-1 \leq x \leq 1$.
 - Find the zeros of $P_0(x), \dots, P_5(x)$.
25. It can be shown that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n - 2k)!}{k!(n - k)!(n - 2k)!} x^{n-2k},$$

⁸Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

Since these limits are finite, $x = 0$ is a regular singular point. For $x = 2$ we have

$$\lim_{x \rightarrow 2} (x-2)p(x) = \lim_{x \rightarrow 2} (x-2) \frac{3}{2(x-2)^2} = \lim_{x \rightarrow 2} \frac{3}{2(x-2)},$$

so the limit does not exist; hence $x = 2$ is an irregular singular point.

EXAMPLE
7

Determine the singular points of

$$\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)y' + (\sin x)y = 0$$

and classify them as regular or irregular.

The only singular point is $x = \pi/2$. To study it we consider the functions

$$\left(x - \frac{\pi}{2}\right) p(x) = \left(x - \frac{\pi}{2}\right) \frac{Q(x)}{P(x)} = \frac{\cos x}{x - \pi/2}$$

and

$$\left(x - \frac{\pi}{2}\right)^2 q(x) = \left(x - \frac{\pi}{2}\right)^2 \frac{R(x)}{P(x)} = \sin x.$$

Starting from the Taylor series for $\cos x$ about $x = \pi/2$, we find that

$$\frac{\cos x}{x - \pi/2} = -1 + \frac{(x - \pi/2)^2}{3!} - \frac{(x - \pi/2)^4}{5!} + \dots,$$

which converges for all x . Similarly, $\sin x$ is analytic at $x = \pi/2$. Therefore, we conclude that $\pi/2$ is a regular singular point for this equation.

PROBLEMS

In each of Problems 1 through 18 find all singular points of the given equation and determine whether each one is regular or irregular.

1. $xy'' + (1-x)y' + xy = 0$
2. $x^2(1-x)^2y'' + 2xy' + 4y = 0$
3. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$
4. $x^2(1-x^2)y'' + (2/x)y' + 4y = 0$
5. $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$
6. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel equation
7. $(x+3)y'' - 2xy' + (1-x^2)y = 0$
8. $x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0$
9. $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$
10. $x(3-x)y'' + (x+1)y' - 2y = 0$
11. $(x^2+x-2)y'' + (x+1)y' + 2y = 0$
12. $xy'' + e^x y' + (3 \cos x)y = 0$
13. $y'' + (\ln|x|)y' + 3xy = 0$
14. $x^2y'' + 2(e^x - 1)y' + (e^{-x} \cos x)y = 0$
15. $x^2y'' - 3(\sin x)y' + (1+x^2)y = 0$
16. $xy'' + y' + (\cot x)y = 0$
17. $(\sin x)y'' + xy' + 4y = 0$
18. $(x \sin x)y'' + 3y' + xy = 0$

In each of Problems 19 and 20 show that the point $x = 0$ is a regular singular point. In each problem try to find solutions of the form $\sum_{n=0}^{\infty} a_n x^n$. Show that there is only one nonzero solution of this form in Problem 19 and that there are no nonzero solutions of this form in Problem 20. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

19. $2xy'' + 3y' + xy = 0$
20. $2x^2y'' + 3xy' - (1+x)y = 0$

21. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point x_0 is finite. In more advanced work in differential equations it is often necessary to discuss the point at infinity. This is done by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$. Show that for the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$ the point at infinity is an ordinary point if

$$\frac{1}{P(1/\xi)} \left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2} \right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4 P(1/\xi)}$$

have Taylor series expansions about $\xi = 0$. Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both

$$\frac{\xi}{P(1/\xi)} \left[\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2} \right] \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2 P(1/\xi)}$$

do have such expansions.

In each of Problems 22 through 27 use the results of Problem 21 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

22. $y'' + y = 0$
 23. $x^2 y'' + x y' - 4y = 0$
 24. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, Legendre equation
 25. $x^2 y'' + x y' + (x^2 - \nu^2)y = 0$, Bessel equation
 26. $y'' - 2xy' + \lambda y = 0$, Hermite equation
 27. $y'' - xy = 0$, Airy equation

5.5 Euler Equations

The simplest example of a differential equation having a regular singular point is the **Euler equation**, or the equidimensional equation,

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0, \quad (1)$$

where α and β are real constants. It is easy to show that $x = 0$ is a regular singular point of Eq. (1). Because the solution of the Euler equation is typical of the solutions of all differential equations with a regular singular point, it is worthwhile considering this equation in detail before discussing the more general problem.

In any interval not including the origin, Eq. (1) has a general solution of the form $y = c_1 y_1(x) + c_2 y_2(x)$, where y_1 and y_2 are linearly independent. For convenience we first consider the interval $x > 0$, extending our results later to the interval $x < 0$. First, we note that $(x^r)' = r x^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$. Hence, if we assume that Eq. (1) has a solution of the form

$$y = x^r, \quad (2)$$

then we obtain

$$\begin{aligned} L[x^r] &= x^2(x^r)'' + \alpha x(x^r)' + \beta x^r \\ &= x^r[r(r-1) + \alpha r + \beta]. \end{aligned} \quad (3)$$

(11), and (18) to obtain real-valued solutions valid in any interval not containing the origin (also see Problems 30 and 31). These results are summarized in the following theorem.

Theorem 5.5.1 The general solution of the Euler equation (1),

$$x^2y'' + \alpha xy' + \beta y = 0,$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r - 1) + \alpha r + \beta = 0.$$

If the roots are real and different, then

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \quad (24)$$

If the roots are real and equal, then

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}. \quad (25)$$

If the roots are complex, then

$$y = |x|^\lambda [c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)], \quad (26)$$

where $r_1, r_2 = \lambda \pm i\mu$.

The solutions of an Euler equation of the form

$$(x - x_0)^2y'' + \alpha(x - x_0)y' + \beta y = 0 \quad (27)$$

are similar to those given in Theorem 5.5.1. If one looks for solutions of the form $y = (x - x_0)^r$, then the general solution is given by either Eq. (24), (25), or (26) with x replaced by $x - x_0$. Alternatively, we can reduce Eq. (27) to the form of Eq. (1) by making the change of independent variable $t = x - x_0$.

The situation for a general second order differential equation with a regular singular point is analogous to that for an Euler equation. We consider that problem in the next section.

PROBLEMS

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

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|------------------------------|--|
| 1. $x^2y'' + 4xy' + 2y = 0$ | 2. $(x + 1)^2y'' + 3(x + 1)y' + 0.75y = 0$ |
| 3. $x^2y'' - 3xy' + 4y = 0$ | 4. $x^2y'' + 3xy' + 5y = 0$ |
| 5. $x^2y'' - xy' + y = 0$ | 6. $(x - 1)^2y'' + 8(x - 1)y' + 12y = 0$ |
| 7. $x^2y'' + 6xy' - y = 0$ | 8. $2x^2y'' - 4xy' + 6y = 0$ |
| 9. $x^2y'' - 5xy' + 9y = 0$ | 10. $(x - 2)^2y'' + 5(x - 2)y' + 8y = 0$ |
| 11. $x^2y'' + 2xy' + 4y = 0$ | 12. $x^2y'' - 4xy' + 4y = 0$ |

In each of Problems 13 through 16 find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as $x \rightarrow 0$.

- ▶ 13. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$
- ▶ 14. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$
- ▶ 15. $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$
- ▶ 16. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$

17. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \rightarrow 0$.
18. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \rightarrow 0$.
19. Find γ so that the solution of the initial value problem $x^2y'' - 2y = 0, y(1) = 1, y'(1) = \gamma$ is bounded as $x \rightarrow 0$.
20. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + (5/2)y = 0$ approach zero as $x \rightarrow \infty$.
21. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that
- All solutions approach zero as $x \rightarrow 0$.
 - All solutions are bounded as $x \rightarrow 0$.
 - All solutions approach zero as $x \rightarrow \infty$.
 - All solutions are bounded as $x \rightarrow \infty$.
 - All solutions are bounded both as $x \rightarrow 0$ and as $x \rightarrow \infty$.
22. Using the method of reduction of order, show that if r_1 is a repeated root of $r(r-1) + \alpha r + \beta = 0$, then x^{r_1} and $x^{r_1} \ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.
23. **Transformation to a Constant Coefficient Equation.** The Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ can be reduced to an equation with constant coefficients by a change of the independent variable. Let $x = e^z$, or $z = \ln x$, and consider only the interval $x > 0$.
- Show that

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}.$$

- Show that the Euler equation becomes

$$\frac{d^2y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0.$$

Letting r_1 and r_2 denote the roots of $r^2 + (\alpha - 1)r + \beta = 0$, show that

- If r_1 and r_2 are real and different, then

$$y = c_1 e^{r_1 z} + c_2 e^{r_2 z} = c_1 x^{r_1} + c_2 x^{r_2}.$$

- If r_1 and r_2 are real and equal, then

$$y = (c_1 + c_2 z) e^{r_1 z} = (c_1 + c_2 \ln x) x^{r_1}.$$

- If r_1 and r_2 are complex conjugates, $r_1 = \lambda + i\mu$, then

$$y = e^{\lambda z} [c_1 \cos(\mu z) + c_2 \sin(\mu z)] = x^\lambda [c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].$$

In each of Problems 24 through 29 use the method of Problem 23 to solve the given equation for $x > 0$.

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|---------------------------------------|---|
| 24. $x^2y'' - 2y = 0$ | 25. $x^2y'' - 3xy' + 4y = \ln x$ |
| 26. $x^2y'' + 7xy' + 5y = x$ | 27. $x^2y'' - 2xy' + 2y = 3x^2 + 2 \ln x$ |
| 28. $x^2y'' + xy' + 4y = \sin(\ln x)$ | 29. $3x^2y'' + 12xy' + 9y = 0$ |
30. Show that if $L[y] = x^2y'' + \alpha xy' + \beta y$, then

$$L[(-x)^r] = (-x)^r F(r)$$

for all $x < 0$, where $F(r) = r(r-1) + \alpha r + \beta$. Hence conclude that if $r_1 \neq r_2$ are roots of $F(r) = 0$, then linearly independent solutions of $L[y] = 0$ for $x < 0$ are $(-x)^{r_1}$ and $(-x)^{r_2}$.

31. Suppose that x^{r_1} and x^{r_2} are solutions of an Euler equation for $x > 0$, where $r_1 \neq r_2$, and r_1 is an integer. According to Eq. (24) the general solution in any interval not containing the origin is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$. Show that the general solution can also be written as