MATH 337 SPRING 2013
HOMEWORK #13
DUE: 2 MAY 2013

1. Practice Questions

- 8.5 # 1-15 (odd)
- 8.6 # 1-11, 17-39 (odd)

2. Assignment Questions

- 8.5 # 2 (5.6.2), 14 (5.6.14) ,
- 8.6 # 2 (5.7.2), 14 (5.7.14), 18 (5.7.18)

integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22); if r_1 and r_2 differ by an integer, this solution corresponds to the larger value of r. If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.7 and 5.8.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of x. However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If P, Q, and R are polynomials, it is often much better to work directly with Eq. (1) than with Eq. (3). This avoids the necessity of expressing $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ as power series. For example, it is more convenient to consider the equation

$$x(1+x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2y'' + \frac{2x}{1+x}y' + \frac{x^2}{1+x}y = 0,$$

which would entail expanding $2x/(1+x)$ and $x^2/(1+x)$ in power series.

PROBLEMS

In each of Problems 1 through 10 show that the given differential equation has a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
3. $xy'' + y = 0$
4. $xy'' + y' - y = 0$
5. $3x^2y'' + 2xy' + x^2y = 0$
6. $x^2y'' + xy' + (x-2)y = 0$
7. $xy'' + (1-x)y' - y = 0$
8. $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$
9. $x^2y'' - x(x+3)y' + (x+3)y = 0$
10. $x^2y'' + (x^2 + \frac{1}{4})y = 0$
11. The Legendre equation of order α is

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

The solution of this equation near the ordinary point $x = 0$ was discussed in Problems 22 and 23 of Section 5.3. In Example 5 of Section 5.4 it was shown that $x = \pm 1$ are regular singular points. Determine the indicial equation and its roots for the point $x = 1$. Find a series solution in powers of $x - 1$ for $x - 1 > 0$.
Hint: Write $1 + x = 2 + (x - 1)$ and $x = 1 + (x - 1)$. Alternatively, make the change of variable $x - 1 = t$ and determine a series solution in powers of t.

12. The Chebyshev equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant; see Problem 10 of Section 5.3.
(a) Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.
(b) Find two linearly independent solutions about $x = 1$.

13. The Laguerre[11] differential equation is

$$xy'' + (1-x)y' + \lambda y = 0.$$

Show that $x = 0$ is a regular singular point. Determine the indicial equation, its roots, the recurrence relation, and one solution ($x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized this polynomial is known as the Laguerre polynomial, $L_m(x)$.

14. The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0.$$

Show that $x = 0$ is a regular singular point; that the roots of the indicial equation are $r_1 = r_2 = 0$; and that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Show that the series converges for all x. The function J_0 is known as the Bessel function of the first kind of order zero.

15. Referring to Problem 14, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.
Hint: If $y_2(x) = J_0(x)v(x)$, then

$$y_2(x) = J_0(x)\int \frac{dx}{x[J_0(x)]^2}.$$

Find the first term in the series expansion of $1/x[J_0(x)]^2$.

16. The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

(a) Show that $x = 0$ is a regular singular point; that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$; and that one solution for $x > 0$ is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!\,n!\,2^{2n}}.$$

Show that the series converges for all x. The function J_1 is known as the Bessel function of the first kind of order one.
(b) Show that it is impossible to determine a second solution of the form

$$x^{-1}\sum_{n=0}^{\infty} b_n x^n, \qquad x > 0.$$

5.7 Series Solutions near a Regular Singular Point, Part II

Now let us consider the general problem of determining a solution of the equation

$$L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{1}$$

[11] Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879.

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either of the intervals $-\rho < x < 0$ or $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \tag{21}$$

where the $a_n(r_1)$ are given by the recurrence relation (8) with $a_0 = 1$ and $r = r_1$.

If $r_1 - r_2$ is not zero or a positive integer, then in either of the intervals $-\rho < x < 0$ or $0 < x < \rho$, there exists a second linearly independent solution of the form

$$y_2(x) = |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right]. \tag{22}$$

The $a_n(r_2)$ are also determined by the recurrence relation (8) with $a_0 = 1$ and $r = r_2$. The power series in Eqs. (21) and (22) converge at least for $|x| < \rho$.

If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n. \tag{23}$$

If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right]. \tag{24}$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, $c_n(r_2)$, and the constant a can be determined by substituting the form of the series solutions for y in Eq. (1). The constant a may turn out to be zero, in which case there is no logarithmic term in the solution (24). Each of the series in Eqs. (23) and (24) converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$.

PROBLEMS

In each of Problems 1 through 12 find all the regular singular points of the given differential equation. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$
2. $x^2y'' - x(2+x)y' + (2+x^2)y = 0$
3. $x(x-1)y'' + 6x^2y' + 3y = 0$
4. $y'' + 4xy' + 6y = 0$
5. $x^2y'' + 3(\sin x)y' - 2y = 0$
6. $2x(x+2)y'' + y' - xy = 0$
7. $x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0$
8. $(x+1)^2y'' + 3(x^2-1)y' + 3y = 0$
9. $x^2(1-x)y'' - (1+x)y' + 2xy = 0$
10. $(x-2)^2(x+2)y'' + 2xy' + 3(x-2)y = 0$
11. $(4-x^2)y'' + 2xy' + 3y = 0$
12. $x(x+3)^2y'' - 2(x+3)y' - xy = 0$

In each of Problems 13 through 17:

(a) Show that $x = 0$ is a regular singular point of the given differential equation.
(b) Find the exponents at the singular point $x = 0$.

(c) Find the first three nonzero terms in each of two linearly independent solutions about $x = 0$.

13. $xy'' + y' - y = 0$
14. $xy'' + 2xy' + 6e^x y = 0$; see Problem 1
15. $x(x-1)y'' + 6x^2 y' + 3y = 0$; see Problem 3
16. $xy'' + y = 0$
17. $x^2 y'' + (\sin x)y' - (\cos x)y = 0$
18. Show that

$$(\ln x)y'' + \tfrac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$. Determine the roots of the indicial equation at $x = 1$. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n (x-1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$. What would you expect the radius of convergence of the series to be?

19. In several problems in mathematical physics (for example, the Schrödinger equation for a hydrogen atom) it is necessary to study the differential equation

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]y' - \alpha\beta y = 0, \qquad \text{(i)}$$

where α, β, and γ are constants. This equation is known as the **hypergeometric equation**.
(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are 0 and $1 - \gamma$.
(b) Show that $x = 1$ is a regular singular point, and that the roots of the indicial equation are 0 and $\gamma - \alpha - \beta$.
(c) Assuming that $1 - \gamma$ is not a positive integer, show that in the neighborhood of $x = 0$ one solution of (i) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots.$$

What would you expect the radius of convergence of this series to be?
(d) Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x \right.$$
$$\left. + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right].$$

(e) Show that the point at infinity is a regular singular point, and that the roots of the indicial equation are α and β. See Problem 21 of Section 5.4.

20. Consider the differential equation

$$x^3 y'' + \alpha x y' + \beta y = 0,$$

where α and β are real constants and $\alpha \neq 0$.
(a) Show that $x = 0$ is an irregular singular point.
(b) By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear, and consequently there is only one formal solution of the assumed form.
(c) Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of β/α show that the formal series solution has a zero radius of convergence, and so does not represent an actual solution in any interval.