Dimensionality Reduction

Refresher:

① Matrix Multiplication: $\begin{bmatrix} 2 & -3 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 10 & -8 \\ 12 & -2 \end{bmatrix}$

Steps: * Let us call the first matrix A & the second one B

* Find the dimensions of A and B:

$A_{[2 \times 2]}$, $B_{[2 \times 2]}$

* Figure out the size of the resultant matrix: $C_{[2 \times 2]}$

* $C = \begin{bmatrix} 2 & -3 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} 10 & -8 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} (20-36) & (-16+6) \\ (70+60) & (-56-10) \end{bmatrix}$

$= \begin{bmatrix} -16 & -10 \\ 130 & -66 \end{bmatrix}$

For any cell (i,j) in C $\longrightarrow$ it is the result of multiplying (.) row i from the first matrix . column j from the 2nd matrix

② Determinant of Square Matrix

* The determinant of a 2*2 Matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
* For larger size matrices, the determinant can be obtained using minors and cofactors; a minor for any element is the determinant that results when the row and column are deleted. The cofactor for any element is either the minor or the opposite of it (if the row & column #s add up to an even no., then it is +ve)
* Steps to find determinant:
  - a) Pick any row or column in the matrix
  - b) Multiply every element in that row or column by its cofactor and add. The result is the determinant.

* **Ex:** $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} - 3 \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 17$

* Applications & Properties:
  - → Det is used in finding the inverse of a matrix :)
  - → If Det = 0, then the matrix is singular & non-invertible
  - → Det are not a matrix
  - → helpful in finding eigen values & eigen vectors

11.1 Eigenvalues and Eigenvectors:

*Defn: Let $\underline{M}$ be a square matrix. Let λ be a constant and $\underline{e}$ a nonzero column vector with the same number of rows as M. Then λ is an eigenvalue of $\underline{M}$ if $\underline{M}\underline{e} = \lambda\underline{e}$

* Here, we assume (let) every eigenvector to be a unit vector (i.e. sum of squares of the components of the vector is 1).
* The first non zero component of the eigenvector to be +ve.

→ in this class not a true condition

Exercise:
Let $M = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$. Validate if $\underline{e} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\lambda = 7$, then these correspond to an eigenvector & its corresponding eigenvalue of $\underline{M}$?

11.2 Computing Eigenvalues and Eigenvectors

* Using software ⟶ Fairly straightforward
  Ex: Using Matlab, [E, Lamda] = eigs(M); [Note, we can use eig]

* Old school way:

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Note: $\underline{Me} = \lambda\underline{e}$ is the same as $(\underline{M} - \lambda\underline{I})\underline{e} = \underline{0}$ ⇐

① $M - \lambda I = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{bmatrix}$

② $|M - \lambda I| = (3-\lambda)(6-\lambda) - 4 = 0$

$$18 + \lambda^2 - 9\lambda - 4 = 0$$
$$\lambda^2 - 9\lambda + 14 = 0$$
$$(\lambda - 7)(\lambda - 2) = 0$$

∴ $\lambda = 7$ or $\lambda = 2$

→ principal eigenvalue ∵ it is larger

③ Let $\underline{e}$ be the vectors of unknowns $= \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 3x + 2y \\ 2x + 6y \end{bmatrix} = \begin{bmatrix} 7x \\ 7y \end{bmatrix}$$

Both of these two equations, say the same thing $y = 2x$, thus a possible eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- Note that this vector is not a unit vector ∵ the sum of its components squared = 5 ∴ if we divide by $\sqrt{5}$, we will get the eigen vector in the Exercise above

- For the 2nd eigenpair repeat steps with 2 instead of 7 :)

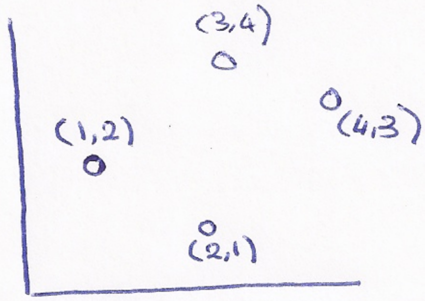
11.4 The Matrix of Eigenvectors: (output from Matlab)

- Suppose that we have an n by n matrix $\underline{M}$, whose eigenvectors are viewed as column vectors $(\underline{e}_1, \underline{e}_2, \cdots \underline{e}_n)$. Let $\underline{E}$ be the matrix whose ith column is $\underline{e}_i$.
- Cool property: $\underline{E}\,\underline{E}^T = \underline{E}^T\underline{E} = \underline{I}$ → eigenvectors of a matrix are orthonormal

11.2 PCA:

* It is a technique for taking a dataset and find new axes that "best describe the data" $\Longrightarrow$ projection of data onto the most important axes; the ones corresponding to the largest eigenvalues :)

An Example:



1] Put the data in a Matrix form

$$M = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} \end{array}$$

2] Compute $M^T M = A$ — matrix of eigen vectors for this is PCA

$$= \underset{(2\times 4)}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}} \times \underset{(4\times 2)}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}}$$

3] Compute $|A - \lambda I|$

$$A - \lambda I = \begin{bmatrix} 30-\lambda & 28 \\ 28 & 30-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = (30-\lambda)(30-\lambda) - 28^2$$
$$= 900 + \lambda^2 - 60\lambda - 28^2$$

4] $|A - \lambda I| = 0$

$\hookrightarrow \lambda = 58 \quad , \quad \lambda = 2$

5] $$\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 58 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left.\begin{array}{l} 30x + 28y = 58x \\ 28x + 30y = 58y \end{array}\right] \longrightarrow \text{Solve these 2 equations together } (x = y)$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow$ unit eigen vector is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Solving for the 2nd eigenpair, we get $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore$ the matrix of eigenvectors, E, $= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

6] If you multiply $ME = \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{2} & -1/\sqrt{2} \\ 7/\sqrt{2} & 1/\sqrt{2} \\ 7/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$