Portfolio Optimization with Many Risky Assets

BKM 6.4

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Portfolio Returns

Suppose you can now invest in an arbitrary number (N) of risky assets.

- Index the assets by $i = 1, \ldots, N$.
- Let ω_i be the fraction of income invested in asset i.
- We will always assume that $\sum_{i=1}^{N} \omega_i = 1$.
- We will denote the return to asset i by r_i.
- The portfolio return is expressed as

$$r_p = \sum_{i=1}^{N} \omega_i r_i.$$

Portfolio Moments

From the properties of expectation and variance, we can compute the mean and variance of the portfolio return.

- Recognize that the N asset returns, r_i, are random variables.
- Denote the means of r_i as μ_i.
- The $N \times N$ covariance matrix of the returns contains the variances, σ_i^2, and covariances, $\text{Cov}\left(r_i, r_j\right) = \sigma_{ij}$:

$$\Sigma_P = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2 \end{bmatrix}$$

Portfolio Moments

Thus resulting moments of the portfolio are

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i$$
$$\sigma_p^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \omega_i \omega_j \sigma_{ij}.$$

What are other ways to express σ_p^2?

Optimization: Risky Minimum-Variance Frontier

To determine the set of efficient risky portfolios (the risky frontier), the investor solves

$$\min_{\{\omega_i\}_{i=1}^{N-1}} \sigma_P^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \omega_i \omega_j \sigma_{ij} \tag{1}$$

subject to

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i \tag{2}$$

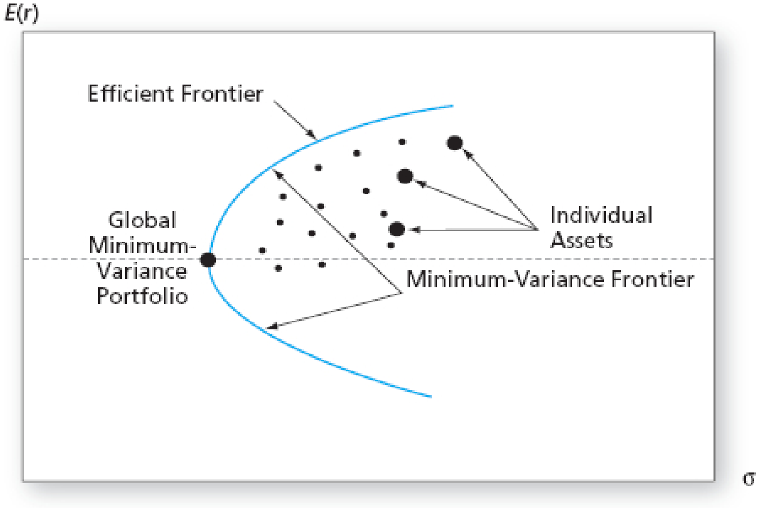
where μ_p is some prespecified value of the portfolio mean return.

Optimization: Risky Minimum-Variance Frontier

Note that

- The optimization problem has $N-1$ choice variables: $\{\omega_i\}_{i=1}^{N-1}$.
- ω_N is not a choice variable because it is found from the constraint: $\omega_N = 1 - \sum_{i=1}^{N-1} \omega_i$.
- This is a challenging problem that is only tractable with linear algebra (we won't solve it).

Risky Minimum-Variance Frontier

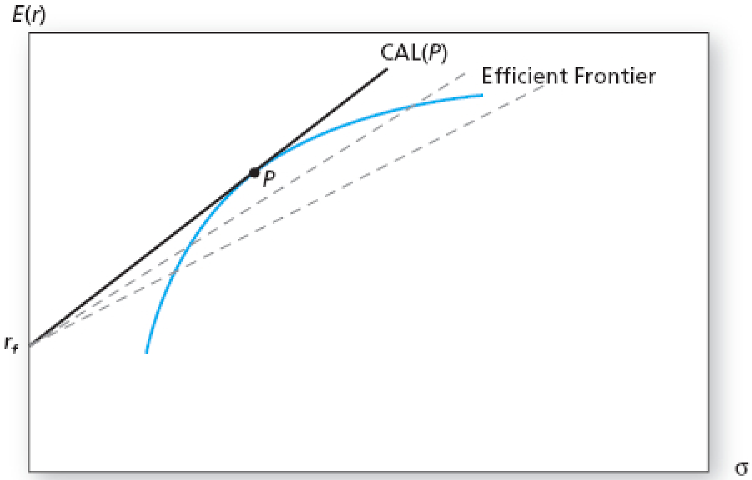


Risky Minimum-Variance Frontier

The frontier generated by multiple risky assets is known as the risky minimum-variance (MV) frontier.

- The lower portion of the frontier is inefficient since a higher mean portfolio exists with the same volatility on the upper portion of the frontier.
- The efficient MV frontier is generated by allowing investment in a risk-free asset and finding the CAL which is tangent to the risky efficient MV frontier.

Efficient Minimum-Variance Frontier



Optimization: Efficient Minimum-Variance Frontier

To determine the tangency portfolio, the investor solves the same problem as before

$$\max_{\mu_p,\sigma_p} \mathrm{SR}_p = \frac{\mu_p - r_f}{\sigma_p}$$

subject to

$$\mu_p = \sum_{i=1}^{N} \omega_i \mu_i$$

$$\sigma_p = \sqrt{\sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \omega_i \omega_j \sigma_{ij}}.$$

Optimization: Investor Choice

So far we have specified two optimization problems:

1. To determine the risky minimum-variance frontier by minimizing variance subject to a particular expected return.
2. To determine the tangency portfolio, by maximizing the Sharpe Ratio subject to constraints on the mean and standard deviation.

Neither of these made use of preferences. A final optimization problem would be the same as before:

3. Maximize utility, $U(\mu_p, \sigma_p)$, subject to investing in the tangency portfolio and a risk-free asset.

Estimation

In practice we must estimate μ_i, σ_i^2 and σ_{ij} for $i = 1, \ldots, N$ and $j = i+1, \ldots, N$.

- A total of N estimates of means.
- How many variances and covariances must we estimate?
- A total of N elements on the diagonal (variances).
- All of the elements above *or* below the diagonal (*not both* because of symmetry).
- This is a total of

$$N + (N-1) + (N-2) + \ldots + 2 + 1 = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}.$$

Estimation

The total number of estimates is

$$N + \frac{N(N+1)}{2} = \frac{N(N+3)}{2}.$$

- As an example, a portfolio of 50 stocks requires $\frac{50\times 53}{2} = 1325$ estimates.
- The models of subsequent lectures will reduce this estimation burden.

Portfolio Optimization Recipe

For an arbitrary number, N, of risky assets:

1. Specify (estimate) the return characteristics of all securities (means, variances and covariances).
2. Establish the optimal risky portfolio.
   - Calculate the weights for the tangency portfolio.
   - Compute mean and std. deviation of the tangency portfolio.
3. Allocate funds between the optimal risky portfolio and the risk-free asset.
   - Calculate the fraction of the complete portfolio allocated to the tangency portfolio and to the risk-free asset.
   - Calculate the share of the complete portfolio invested in each asset of the tangency portfolio.

Separation Property

All investors hold some combination of the same two assets: the risk-free asset and the tangency portfolio.

- The optimal risky (tangency portfolio) is the same for all investors, regardless of preferences.
- The tangency portfolio is simply determined by estimation and a mathematical formula.
- Individual preferences determine the exact proportions of wealth each investor will allocate to the two assets.
- This is known as *The Separation Property* or *Two Fund Separation*.

Separation Property

The separation property implies that portfolio choice can be separated into two independent steps:

- Determining the optimal risky portfolio (preference independent).
- Deciding what proportion of wealth to invest in the risk-free asset and the tangency portfolio (preference dependent).

Separation Property

The separation property will not hold if

- Individuals produce different estimates of asset return characteristics (since differing estimates will result in different tangency portfolios).
- Individuals face different constraints (short-sale, tax, etc.).

The Power of Diversification

Let's formalize the benefits of diversification. The variance of a portfolio of N risky assets is

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i\omega_j\sigma_{ij} = \sum_{i=1}^{N}\omega_i^2\sigma_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\omega_i\omega_j\sigma_{ij}.$$

In the case of an equally weighted portfolio,

$$\begin{aligned}\sigma_p^2 &= \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{2}{N^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sigma_{ij} \\ &= \frac{1}{N}\overline{\text{Var}} + \frac{N-1}{N}\overline{\text{Cov}}.\end{aligned}$$

The Power of Diversification

Where

$$\overline{\text{Var}} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2$$

and

$$\overline{\text{Cov}} = \frac{2}{N(N-1)}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\sigma_{ij}.$$

These are the average variance and covariance.

The Power of Diversification

The limit of portfolio variance is

$$\lim_{N\to\infty} \sigma_p^2 = \lim_{N\to\infty} \frac{1}{N}\overline{\text{Var}} + \lim_{N\to\infty} \frac{N-1}{N}\overline{\text{Cov}} = \overline{\text{Cov}}.$$

- If the assets in the portfolio are uncorrelated or not correlated *on average* ($\overline{\text{Cov}} = 0$), there is no limit to diversification: $\sigma_p^2 = 0$.
- If there are systemic sources of risk that affect all assets ($\overline{\text{Cov}} > 0$) there will be a lower bound on ability to diversify: $\sigma_p^2 > 0$.