The Term Structure of Interest Rates

BKM 10.6

Eric M. Aldrich

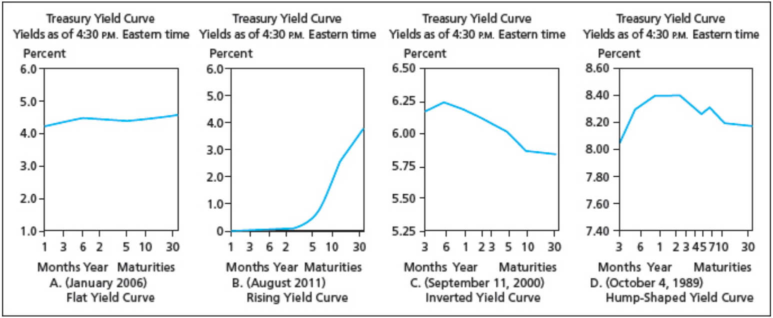
Econ 133

UC Santa Cruz

Yield Curve

Bonds of different maturities often have different yields to maturity.

- The relationship between yield and maturity is summarized graphically in the *yield curve*.
- Consider several examples below.



Yield Curve Slope

An upward sloping yield curve is evidence that short-term interest rates are going to rise.

- Consider two investment strategies.
  - Buy and hold a two-year zero-coupon bond, offering 6% return each year.
  - Buy a one-year bond today, offering a 5% return over the coming year, and roll the investment into another one-year zero-coupon bond a year from now, offering an interest rate of r_2.
- These investments should be equivalent. Why?

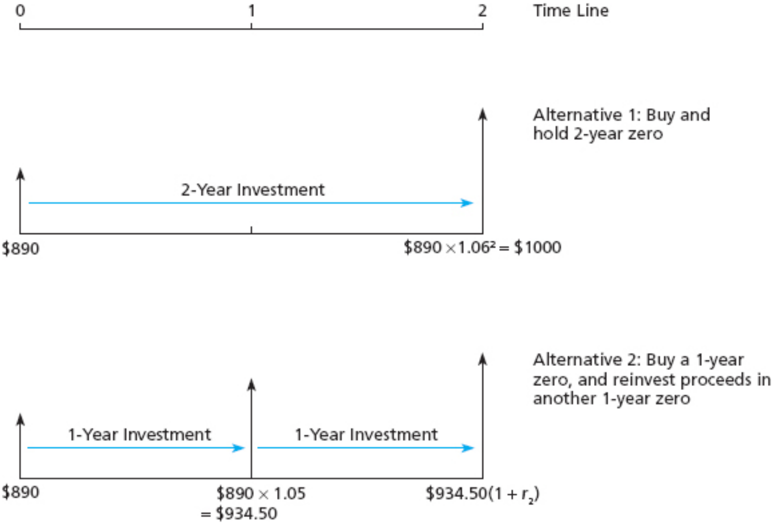
Yield Curve Slope

Suppose you begin with $890 to invest (the price of a two-year zero-coupon bond with 6% YTM.

- Equating the returns to each strategy gives:

$$
\begin{aligned}
\$890 \times (1.06)^2 &= \$890 \times (1.05) \times (1 + r_2) \\
\Rightarrow 1 + r_2 &= \frac{1.06^2}{1.05} = 1.0701 \\
\Rightarrow r_2 &= 0.0701.
\end{aligned}
$$

Two Investment Strategies



Spot Rates and Short Rates

We distinguish between two types of interest rates.

- Spot rate: the rate offered *today* on zero-coupon bonds of different maturities.
  - In the previous example, the one-year spot rate is 5% and the two year spot rate is 6%.
- Short rate: the rate for given time interval (one year) offered at different points in time.
  - In the previous example, the first-year short rate is 5% (same as the spot!) and the second-year short rate is 7.01%.

Spot Rates and Short Rates

The spot rate for a given period should be the geometric average of short rates over that interval.

- Let y_2 be the two-year spot rate.
- Let r_1 and r_2 be the first-year and second-year short rates.
- Don't forget that $y_1 = r_1$.

$$(1 + y_2)^2 = (1 + r_1) \times (1 + r_2)$$
$$\Rightarrow 1 + y_2 = \sqrt{(1 + r_1) \times (1 + r_2)}.$$

- So, if the yield curve slopes up ($y_2 > y_1 = r_1$), we conclude that short-term rates will rise ($r_2 > r_1$).
- Reverse reasoning holds for a downward sloping yield curve.

Spot Rate and Short Rate Example

Assume the following spot rates and short rates:

- Spots: $y_1 = 0.05$, $y_2 = 0.06$ and $y_3 = 0.07$.
- Shorts: $r_1 = y_1$ and $r_2 = 0.0701$.
- What is the three-year short rate, r_3?
- Buying a three-year zero-coupon bond should be identical to buying a two-year zero and rolling into a one-year zero.

$$
\begin{aligned}
(1 + y_3)^3 &= (1 + y_2)^2 \times (1 + r_3) \\
\Rightarrow 1.07^3 &= 1.06^2 \times (1 + r_3) \\
\Rightarrow r_3 &= \frac{1.07^3}{1.06^2} - 1 = 0.09025.
\end{aligned}
$$

Spot Rate and Short Rate Example

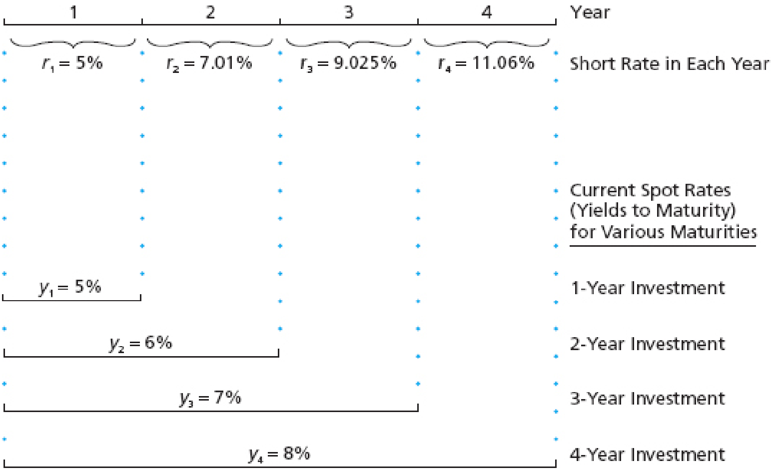
We know

$$(1 + y_2)^2 = (1 + r_1) \times (1 + r_2).$$

So the full decomposition is

$$\begin{aligned}(1 + y_3)^3 &= (1 + y_2)^2 \times (1 + r_3) \\ &= (1 + r_1) \times (1 + r_2) \times (1 + r_3) \\ \Rightarrow 1.07^3 &= 1.05 \times 1.0701 \times 1.09025.\end{aligned}$$

Spot Rate and Short Rate Example



General Short Rates

We can generalize the previous results.

- Investing in an n period zero-coupon bond should be the same as investing in an $n-1$ zero and rolling into a one-period zero at time $n-1$.

$$
\begin{aligned}
(1+y_n)^n &= (1+y_{n-1})^{n-1} \times (1+r_n) \\
\Rightarrow 1+r_n &= \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}}.
\end{aligned}
$$

Forward Rates

In the development above, we assumed no uncertainty.

- All future rates were known at time zero.
- In reality, we don't have perfect knowledge of time n short rates at time zero.
- To distinguish between actual short rates that occur in the future, we define the forward rate to be

$$\Rightarrow 1 + f_n = \frac{(1 + y_n)^n}{(1 + y_{n-1})^{n-1}}.$$

- The time $t = n$ forward rate is the break-even interest rate that equates the returns of an n-period zero-coupon bond with an $(n - 1)$-period zero rolled into a one-period zero.
- It may not be equal to the expected future short rate.

Expectations Hypothesis

The *Expectations Hypothesis* of the yield curve says that expected short rates equal forward rates:

$$\begin{aligned} \mathrm{E}\left[r_n\right] &= f_n \\ \Rightarrow (1+y_n)^n &= (1+y_{n-1})^{n-1}(1+\mathrm{E}\left[r_n\right]). \end{aligned}$$

- If the yield curve slopes upward, short rates are expected to rise: $\mathrm{E}\left[r_n\right] > \mathrm{E}\left[r_{n-1}\right] > r_1 = y_1$.
- If the yield curve slopes downward, short rates are expected to fall: $\mathrm{E}\left[r_n\right] < \mathrm{E}\left[r_{n-1}\right] < r_1 = y_1$.

Liquidity Preference Theory

According to the *Liquidity Preference Theory* of the yield curve, investors must be compensated for holding longer-term bonds.

- Longer-term bonds are subject to greater risk, and so investors should demand a premium for holding them.
- In reality, a *premium* means that investors will only buy them for a lower price (which means greater yield).

Liquidity Preference Theory

The Liquidity Preference Theory can be expressed as forward rates being equal to expected short rates plus a premium, ϕ:

$$f_n = \mathrm{E}\left[r_n\right] + \phi$$
$$\Rightarrow (1+y_n)^n = (1+y_{n-1})^{n-1}(1+\mathrm{E}\left[r_n\right]+\phi).$$

- According to this theory, expected short rates *can be* constant if the yield curve is upward sloping.
- If the yield curve is downward sloping, expected short rates must be falling. Why?

Liquidity Preference Example

Suppose you buy a two-year bond and that

- Short rates for the next two years are constant at 8%: $r_1 = \mathrm{E}[r_2] = 0.08$.
- The liquidity premium for year two is 1%: $\phi = 0.01$.

What is the yield to maturity of the two year bond?

$$\begin{aligned} (1+y_2)^2 &= (1+r_1)(1+f_2) \\ &= (1+r_1)(1+\mathrm{E}[r_2]+\phi) \\ &= (1.08)(1.09) \\ \Rightarrow 1+y_2 &= \sqrt{1.08 \times 1.09} \\ &= 1.085. \end{aligned}$$

- So the yield curve slopes up $(y_2 > y_1)$ even though expected short rates are constant.

Expectations Hypothesis Example

However, if there is no liquidity premium

$$\begin{aligned}(1+y_2)^2 &= (1+r_1)(1+f_2)\\ &= (1+r_1)(1+\mathrm{E}[r_2])\\ &= (1.08)(1.08)\\ \Rightarrow 1+y_2 &= \sqrt{1.08\times 1.08}\\ &= 1.08.\end{aligned}$$

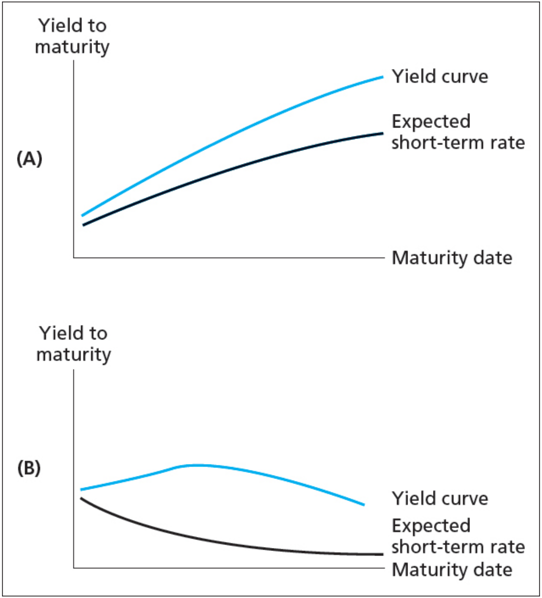
- Now the yield curve is flat.

Implications of the Theories

The slope of the yield curve *always* determines whether *forward rates* are rising or falling.

- $y_2 > y_1$ means $f_2 > f_1$ (by the definition of forward rates!).
- If the Expectations Hypothesis holds, $\mathrm{E}[r_2] = f_2$, so $y_2 > y_1$ means $\mathrm{E}[r_2] > r_1 = f_1 = y_1$.
- If the Liquidity Preference Theory holds, we have no guarantee that $\mathrm{E}[r_2] > r_1$ if $y_2 > y_1$.
  - Short rates could be constant with a moderate liquidity premium.
  - Short rates could be rising some, with a small liquidity premium.
  - Short rates could be falling, with a large liquidity premium.
- What about if $y_2 < y_1$?

Expectations Hypothesis vs. Liquidity Preference Theory



Historical Term Spread