Math 140

Lecture 3

Greg Maloney

with modifications by T. Milev

University of Massachusetts Boston

February 5, 2013

Outline

Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle of a complete rotation contains 360°, which is the same as 2π rad. Therefore

$$\pi \text{ rad} = 180^\circ.$$

Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle of a complete rotation contains 360°, which is the same as 2π rad. Therefore

$$\pi \text{ rad} = 180^\circ.$$

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ, \qquad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}.$$

Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle of a complete rotation contains 360°, which is the same as 2π rad. Therefore

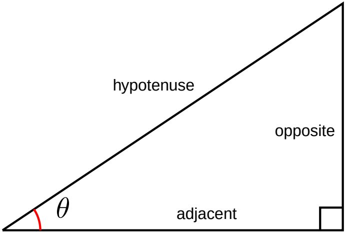
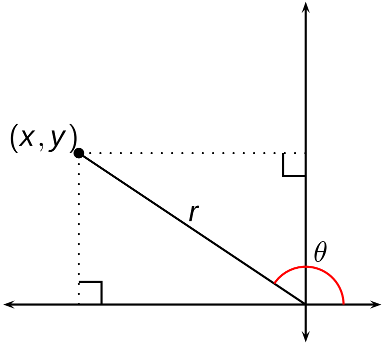
$$\pi \text{ rad} = 180^\circ.$$

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ, \qquad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}.$$

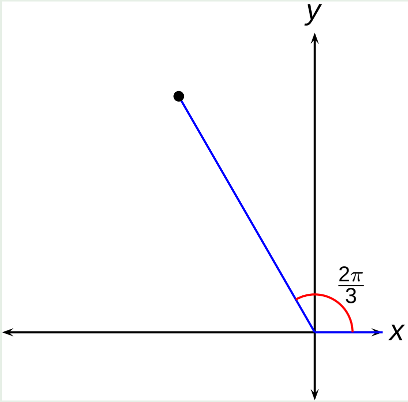
The following table shows the correspondence between degrees and radians for some common angles.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

The Trigonometric Functions

	
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad \csc\theta = \frac{\text{hyp}}{\text{opp}}$	$\sin\theta = \frac{y}{r}$ $\quad \csc\theta = \frac{r}{y}$
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\quad \sec\theta = \frac{\text{hyp}}{\text{adj}}$	$\cos\theta = \frac{x}{r}$ $\quad \sec\theta = \frac{r}{x}$
$\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\quad \cot\theta = \frac{\text{adj}}{\text{opp}}$	$\tan\theta = \frac{y}{x}$ $\quad \cot\theta = \frac{x}{y}$
Acute angles	Obtuse or negative angles

Example

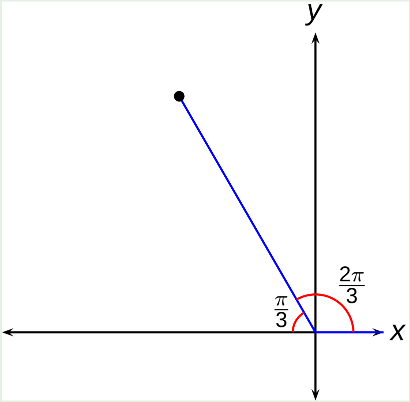


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \qquad \cos\frac{2\pi}{3} = \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

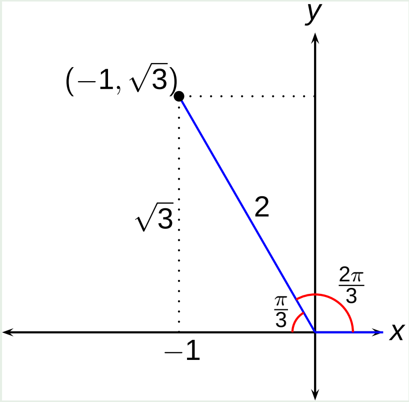


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \qquad \cos\frac{2\pi}{3} = \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

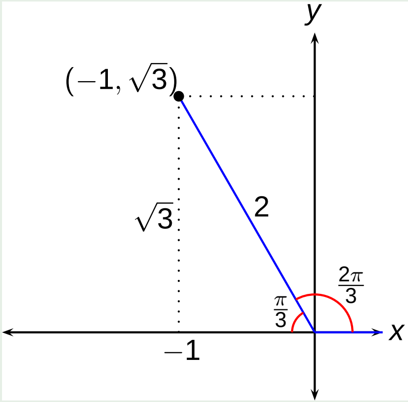


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \qquad \cos\frac{2\pi}{3} = \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

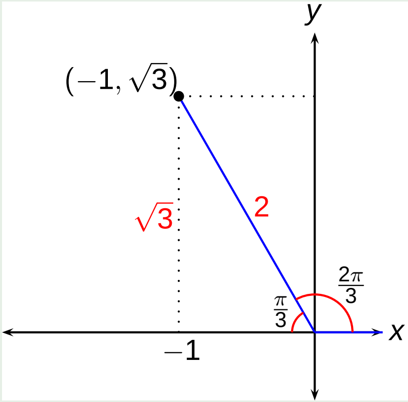


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$\sin \frac{2\pi}{3} =$ $\cos \frac{2\pi}{3} =$ $\tan \frac{2\pi}{3} =$

$\csc \frac{2\pi}{3} =$ $\sec \frac{2\pi}{3} =$ $\cot \frac{2\pi}{3} =$

Example

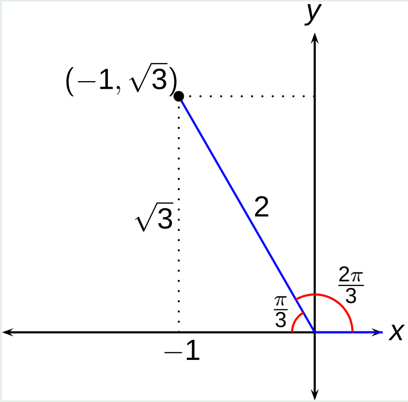


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ $\cos \frac{2\pi}{3} =$ $\tan \frac{2\pi}{3} =$

$\csc \frac{2\pi}{3} =$ $\sec \frac{2\pi}{3} =$ $\cot \frac{2\pi}{3} =$

Example

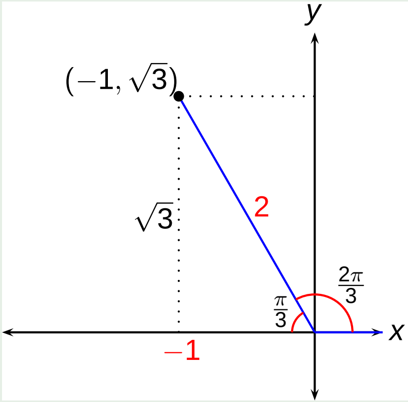


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

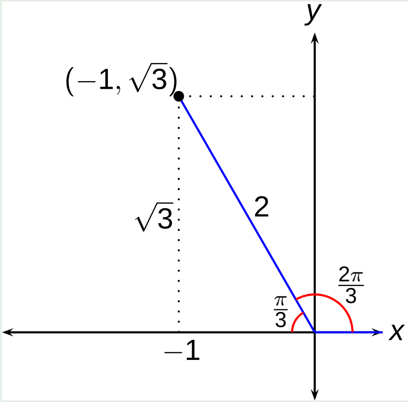


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

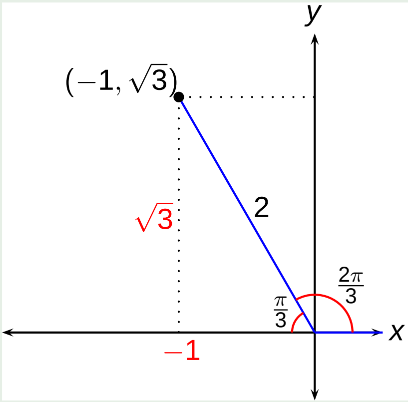


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} =$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

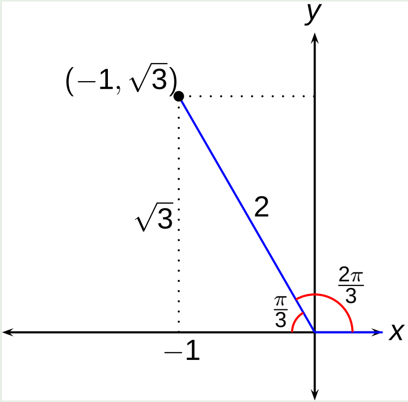


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

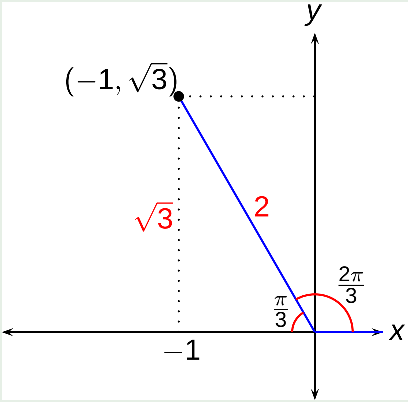


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

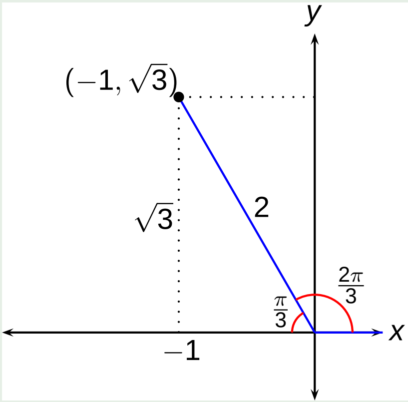


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

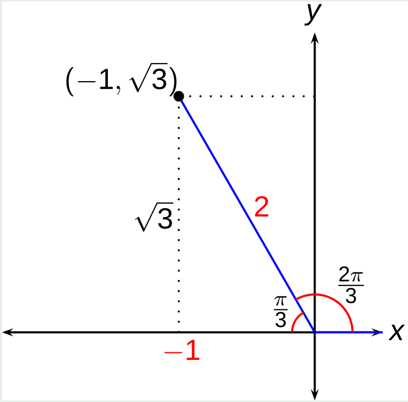


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = \qquad \cot\frac{2\pi}{3} =$$

Example

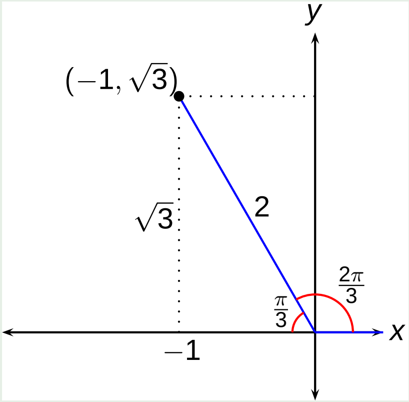


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -\frac{2}{1} = -2 \qquad \cot\frac{2\pi}{3} =$$

Example

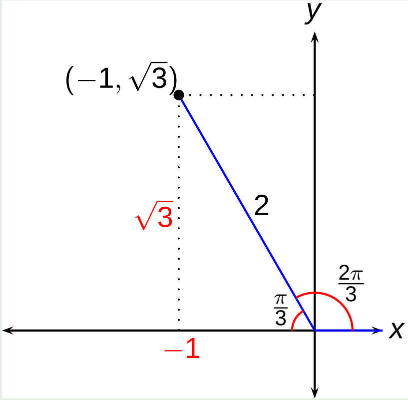


Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -\frac{2}{1} = -2 \qquad \cot\frac{2\pi}{3} =$$

Example



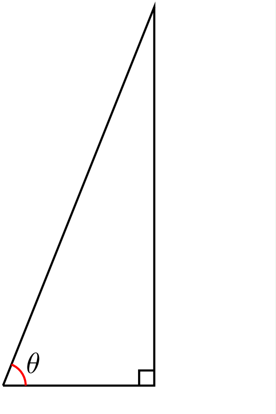
Find the exact trigonometric ratios for $\theta = 2\pi/3 = 120^\circ$.

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -\frac{2}{1} = -2 \qquad \cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



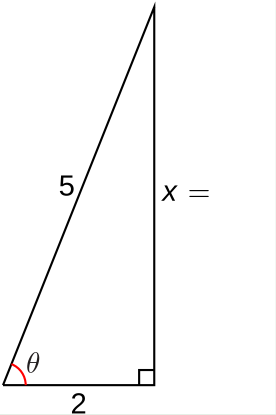
$\sin\theta =$ $\tan\theta =$

$\csc\theta =$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.

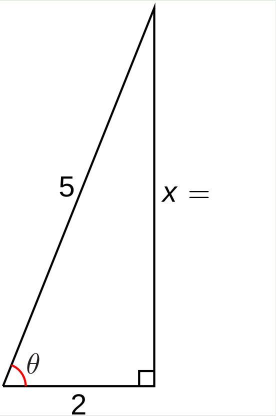
$\sin\theta =$ $\tan\theta =$

$\csc\theta =$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.

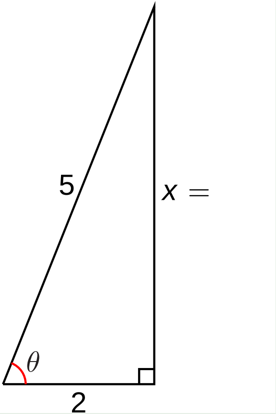
$\sin\theta =$ $\tan\theta =$

$\csc\theta =$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 =$, so $x =$.

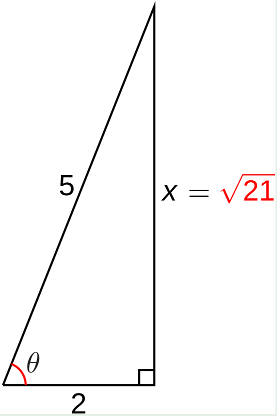
$\sin\theta =$ $\tan\theta =$

$\csc\theta =$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

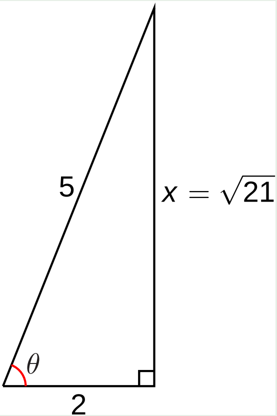
$\sin\theta =$ $\qquad$ $\tan\theta =$

$\csc\theta =$ $\qquad$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

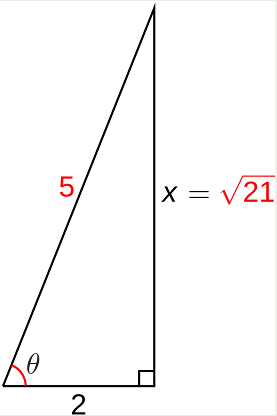
$\sin\theta =$ $\qquad$ $\tan\theta =$

$\csc\theta =$ $\qquad$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

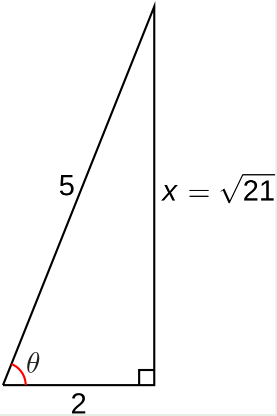
$\sin\theta = \frac{\sqrt{21}}{5}$ $\tan\theta =$

$\csc\theta =$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

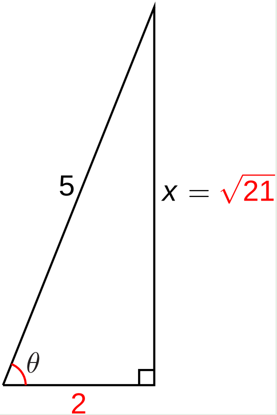
$\sin\theta = \frac{\sqrt{21}}{5}$ $\quad$ $\tan\theta =$

$\csc\theta =$ $\quad$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

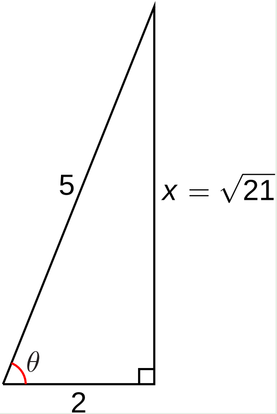
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \qquad \sec\theta =$$

$$\cot\theta =$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

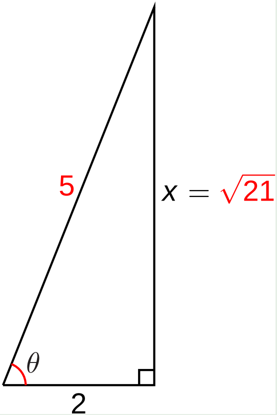
$\sin\theta = \frac{\sqrt{21}}{5}$ $\quad$ $\tan\theta = \frac{\sqrt{21}}{2}$

$\csc\theta =$ $\quad$ $\sec\theta =$

$\cot\theta =$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

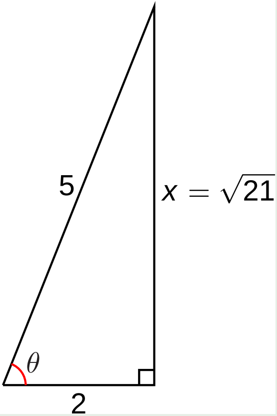
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta =$$

$$\cot\theta =$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

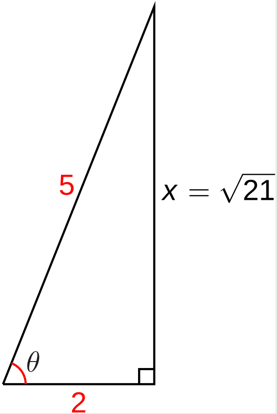
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta =$$

$$\cot\theta =$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

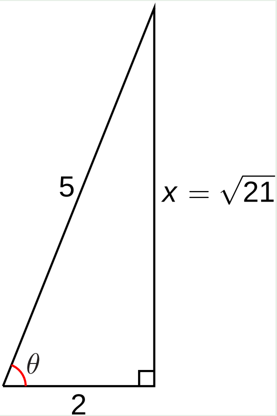
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2}$$

$$\cot\theta =$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

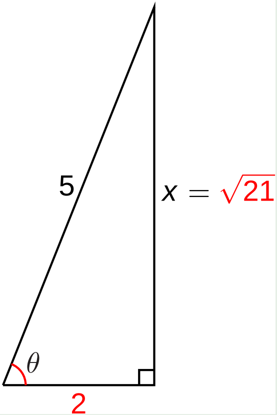
$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2}$$

$$\cot\theta =$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.



- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

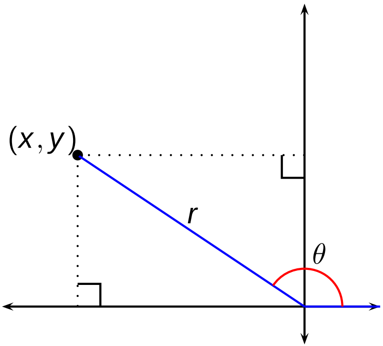
$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2}$$

$$\cot\theta = \frac{2}{\sqrt{21}}$$

Trigonometric Identities

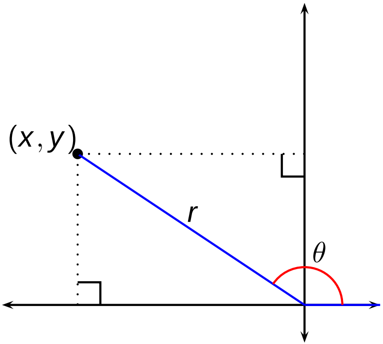
Definition (Trigonometric Identity)

A trigonometric identity is a relationship among the trigonometric functions that is true for any value of the independent variable.

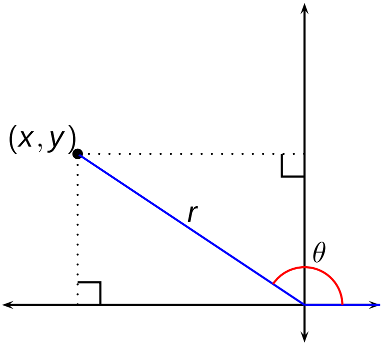


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

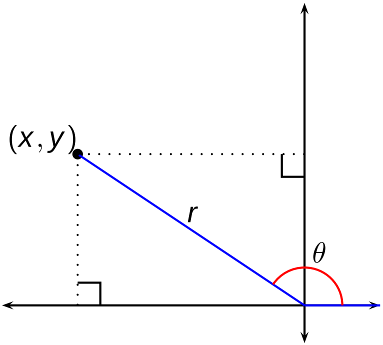


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$



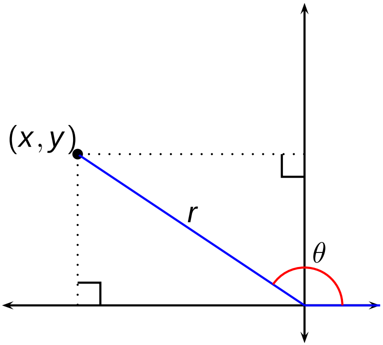
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\sin^2\theta + \cos^2\theta$$



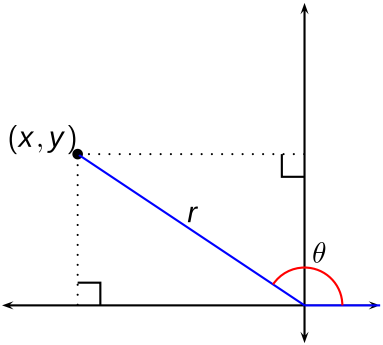
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \quad & \frac{y^2}{r^2} + \frac{x^2}{r^2} \end{aligned}$$



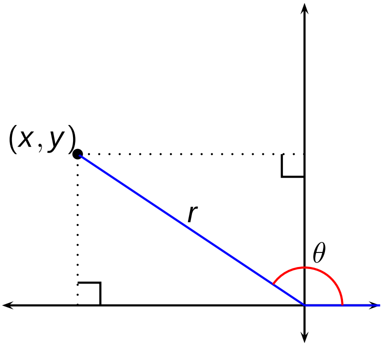
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = & \frac{y^2 + x^2}{r^2} \end{aligned}$$



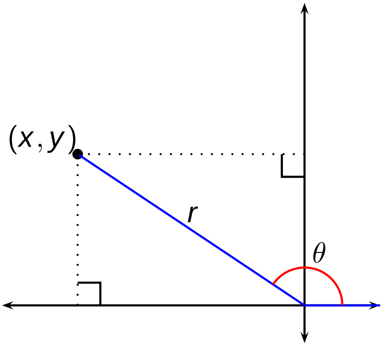
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = & \frac{y^2 + x^2}{r^2} \\ = & \frac{r^2}{r^2} \end{aligned}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

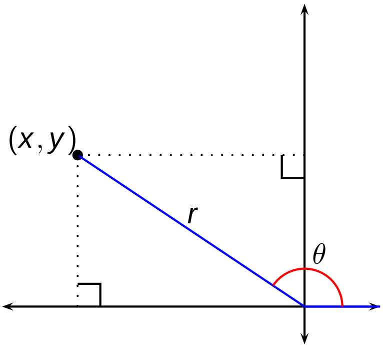
$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \; & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \; & \frac{y^2 + x^2}{r^2} \\ = \; & \frac{r^2}{r^2} \\ = \; & 1 \end{aligned}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \quad & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \quad & \frac{y^2 + x^2}{r^2} \\ = \quad & \frac{r^2}{r^2} \\ = \quad & 1 \end{aligned}$$

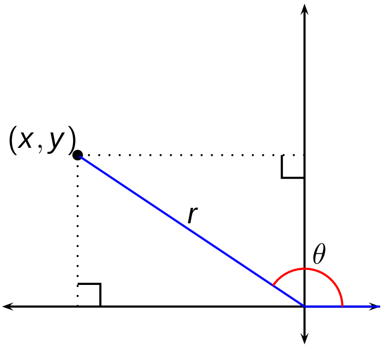
Therefore $\sin^2\theta + \cos^2\theta = 1$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

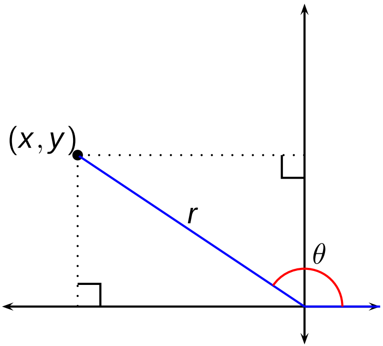


$$\begin{aligned} \sin\theta &= \tfrac{y}{r} & \csc\theta &= \tfrac{r}{y} \\ \cos\theta &= \tfrac{x}{r} & \sec\theta &= \tfrac{r}{x} \\ \tan\theta &= \tfrac{y}{x} & \cot\theta &= \tfrac{x}{y} \end{aligned}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

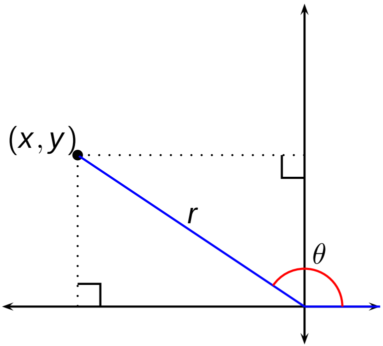


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \end{aligned}$$

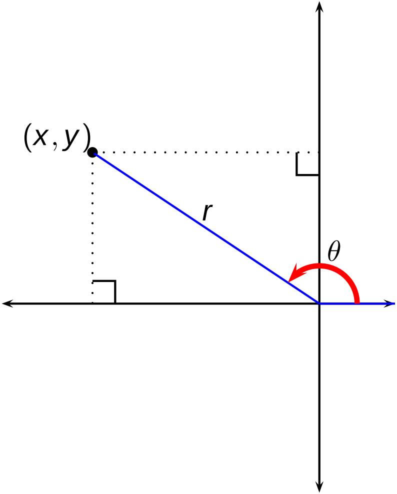


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

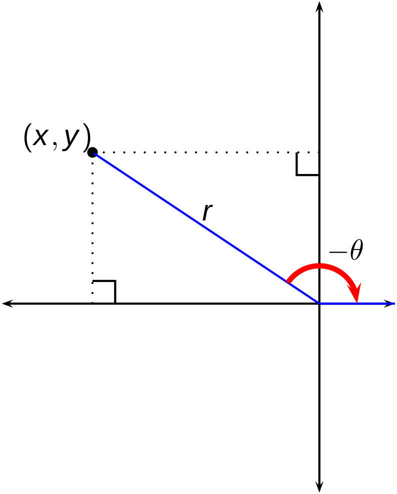
Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$



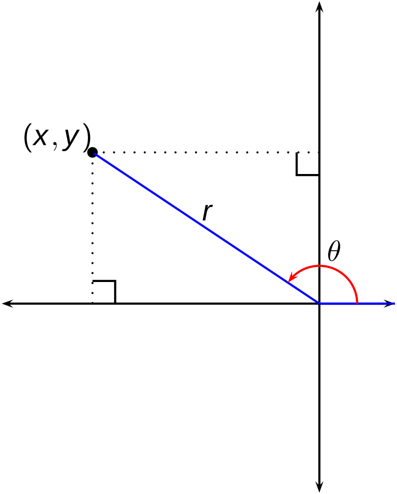
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.



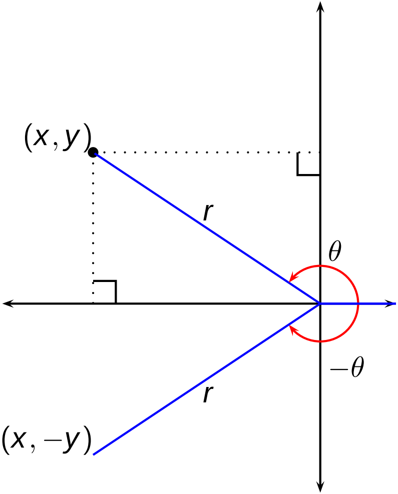
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.



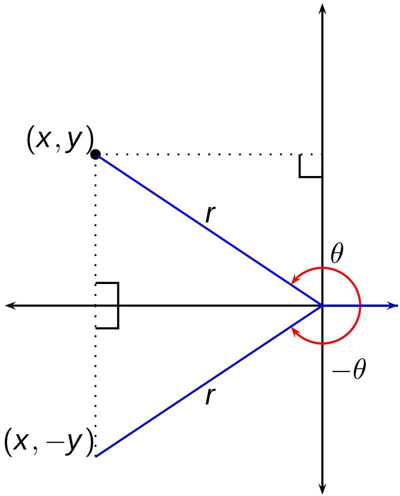
- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.

$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$



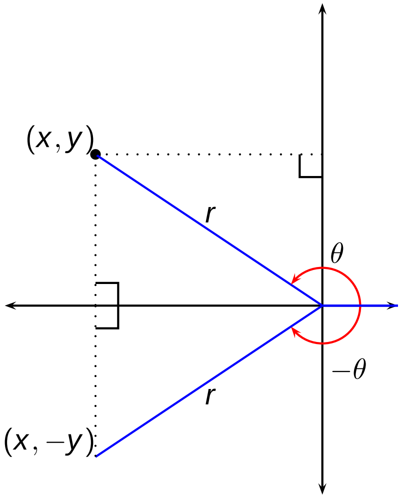
$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.



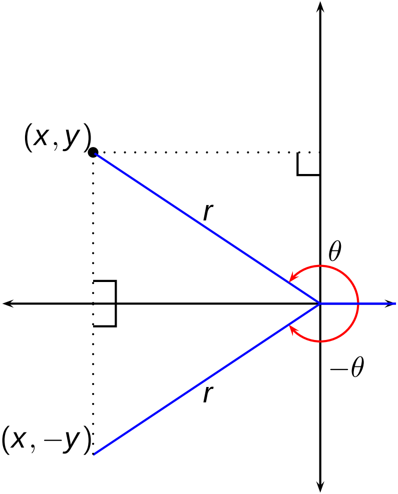
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.



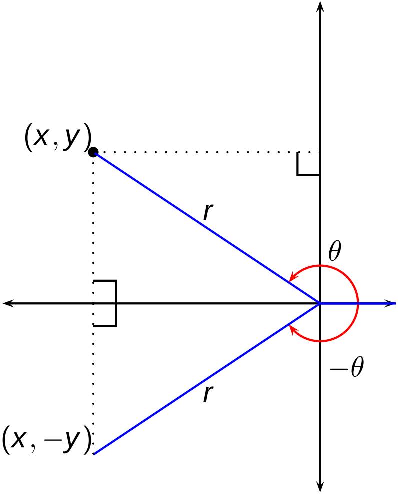
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.



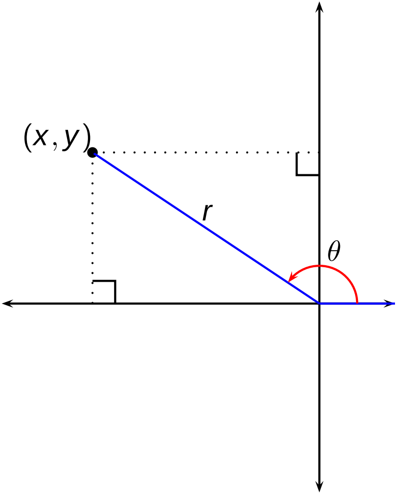
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.

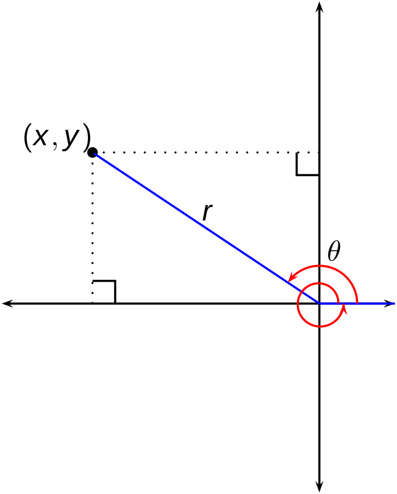


$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.
- cos is an even function.

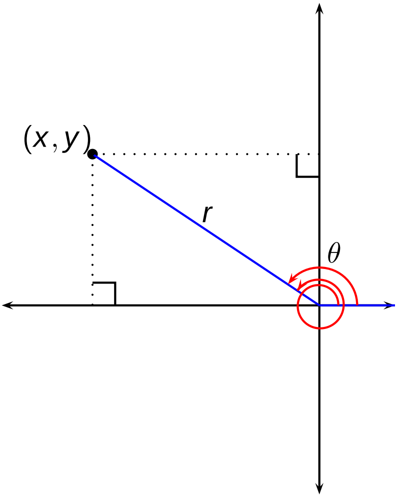


$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$



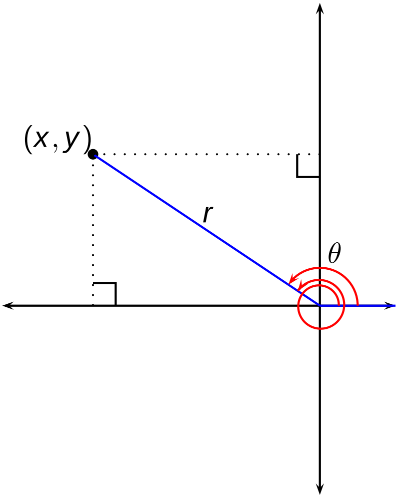
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- 2π represents a full rotation.



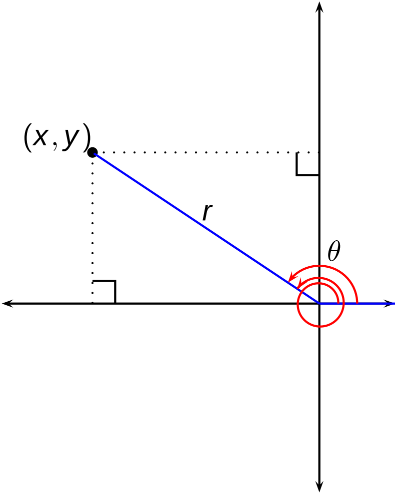
$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.



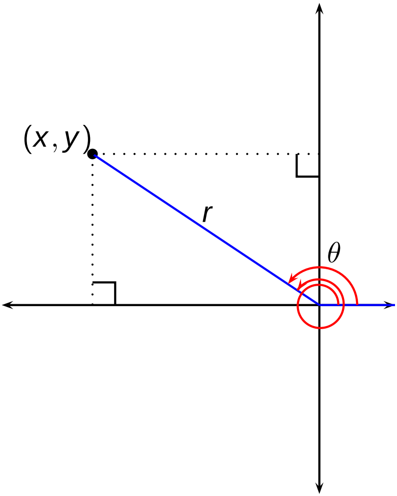
$\sin\theta = \frac{y}{r}$ $\quad\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$ $\quad\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$ $\quad\cot\theta = \frac{x}{y}$

- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.



- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.

$$\begin{array}{ll} \sin\theta = \frac{y}{r} & \csc\theta = \frac{r}{y} \\ \cos\theta = \frac{x}{r} & \sec\theta = \frac{r}{x} \\ \tan\theta = \frac{y}{x} & \cot\theta = \frac{x}{y} \end{array}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say sin and cos are 2π-periodic.

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

Substitute $-y$ for y, and use the fact that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$:

$$\begin{aligned}\sin(x-y) &= \sin x\cos y - \cos x\sin y\\ \cos(x-y) &= \cos x\cos y + \sin x\sin y\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y &+& \cos x\sin y\\ \cos(x+y) &= \cos x\cos y &-& \sin x\sin y\end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ \cos 2x &= \cos^2 x-\sin^2 x\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ \cos 2x &= \cos^2 x - \sin^2 x\end{aligned}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ \cos 2x &= 1 - 2\sin^2 x\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \end{aligned}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}$$

To get the half-angle formulas, solve these equations for $\cos^2 x$ and $\sin^2 x$ respectively.

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

The remaining identities are consequences of the addition formulas:

$$\begin{array}{rclcl} \sin(x+y) & = & \sin x \cos y & + & \cos x \sin y \\ \cos(x+y) & = & \cos x \cos y & - & \sin x \sin y \end{array}$$

The remaining identities are consequences of the addition formulas:

$$\begin{array}{rcccc} \sin(x+y) & = & \sin x \cos y & + & \cos x \sin y \\ \cos(x+y) & = & \cos x \cos y & - & \sin x \sin y \end{array}$$

Divide the first equation by the second, and then cancel $\cos x \cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

Divide the first equation by the second, and then cancel $\cos x\cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x+\tan y}{1-\tan x\tan y}$$

Do the same for the subtraction formulas:

$$\tan(x-y) = \frac{\tan x-\tan y}{1+\tan x\tan y}$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\sin x = \sin 2x$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\sin x = \sin 2x$$

$$\sin x = 2 \sin x \cos x$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned} \sin x &= \sin 2x \\ \sin x &= 2 \sin x \cos x \\ 0 &= 2 \sin x \cos x - \sin x \end{aligned}$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned}
\sin x &= \sin 2x \\
\sin x &= 2\sin x \cos x \\
0 &= 2\sin x \cos x - \sin x \\
0 &= \sin x(2\cos x - 1)
\end{aligned}$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned}
\sin x &= \sin 2x \\
\sin x &= 2\sin x \cos x \\
0 &= 2\sin x \cos x - \sin x \\
0 &= \sin x(2\cos x - 1)
\end{aligned}$$

$$\sin x = 0 \qquad\qquad 2\cos x - 1 = 0$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned} \sin x &= \sin 2x \\ \sin x &= 2 \sin x \cos x \\ 0 &= 2 \sin x \cos x - \sin x \\ 0 &= \sin x (2 \cos x - 1) \end{aligned}$$

$$\sin x = 0 \qquad\qquad 2 \cos x - 1 = 0$$

$$x = 0, \pi, 2\pi$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned} \sin x &= \sin 2x \\ \sin x &= 2\sin x \cos x \\ 0 &= 2\sin x \cos x - \sin x \\ 0 &= \sin x(2\cos x - 1) \end{aligned}$$

$$\sin x = 0 \qquad\qquad 2\cos x - 1 = 0$$

$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \frac{1}{2}$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned} \sin x &= \sin 2x \\ \sin x &= 2\sin x \cos x \\ 0 &= 2\sin x \cos x - \sin x \\ 0 &= \sin x(2\cos x - 1) \end{aligned}$$

$$\sin x = 0 \qquad\qquad 2\cos x - 1 = 0$$

$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Example

Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

$$\begin{aligned} \sin x &= \sin 2x \\ \sin x &= 2 \sin x \cos x \\ 0 &= 2 \sin x \cos x - \sin x \\ 0 &= \sin x (2 \cos x - 1) \end{aligned}$$

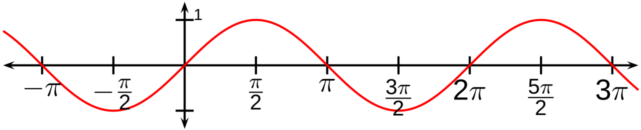
$$\sin x = 0 \qquad\qquad 2\cos x - 1 = 0$$

$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \frac{1}{2}$$

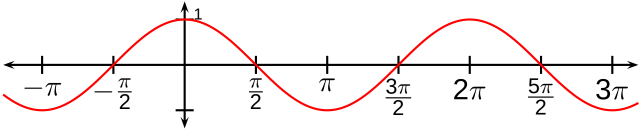
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore the equation has five solutions: 0, $\frac{\pi}{3}$, π, $\frac{5\pi}{3}$, and 2π.

Graphs of the Trigonometric Functions

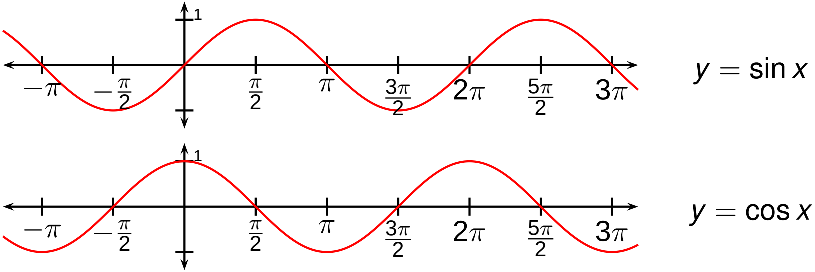


$y = \sin x$



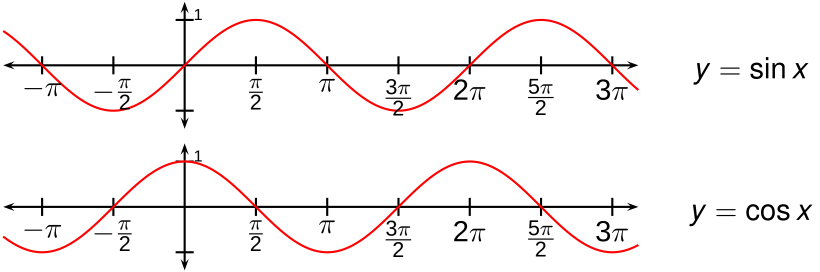
$y = \cos x$

Graphs of the Trigonometric Functions



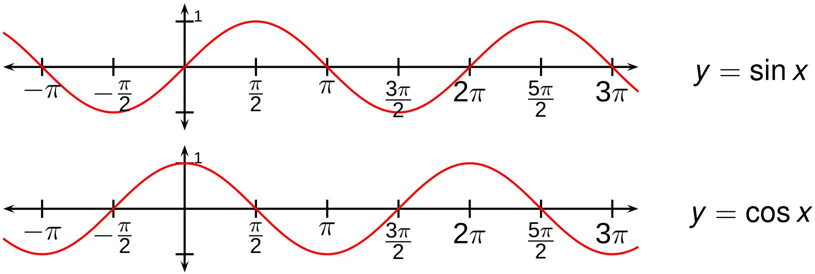
- $\sin x$ has zeroes at $n\pi$ for all integers n.

Graphs of the Trigonometric Functions



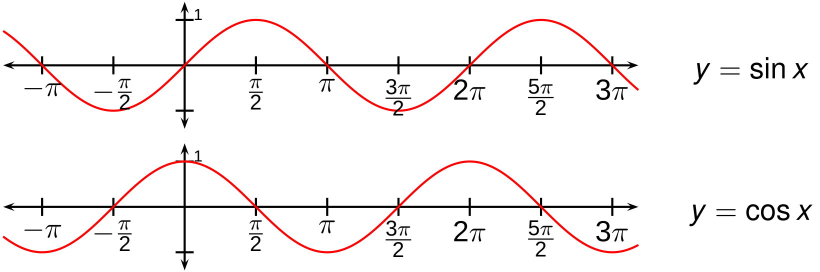
- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\pi/2 + n\pi$ for all integers n.

Graphs of the Trigonometric Functions

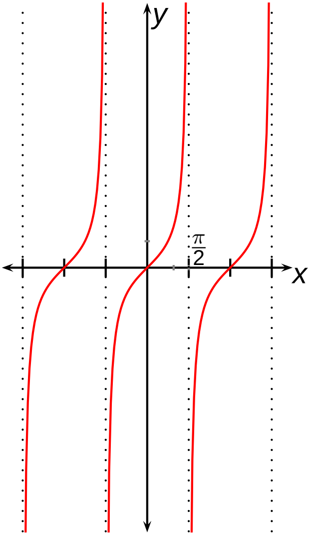


- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\pi/2 + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.

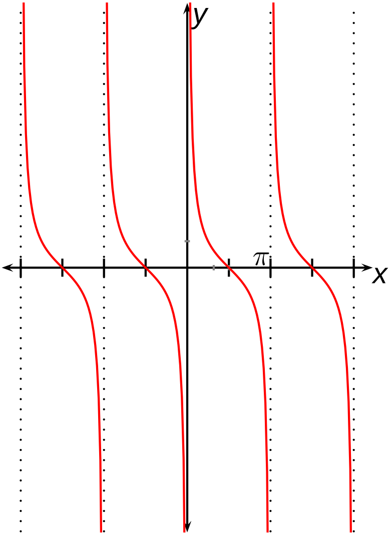
Graphs of the Trigonometric Functions



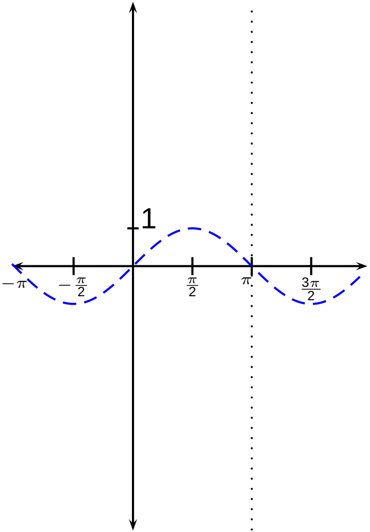
- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\pi/2 + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.
- $-1 \leq \cos x \leq 1$.



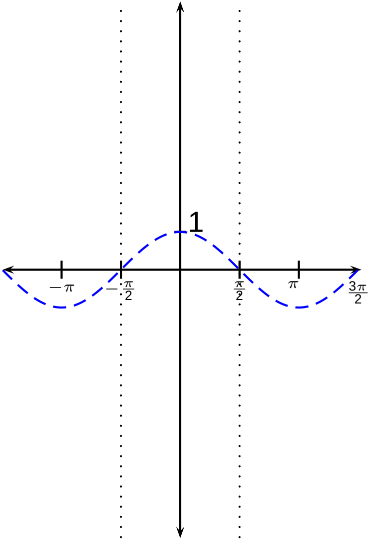
$y = \tan x$



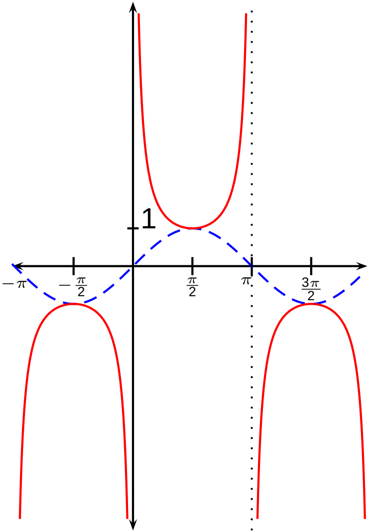
$y = \cot x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

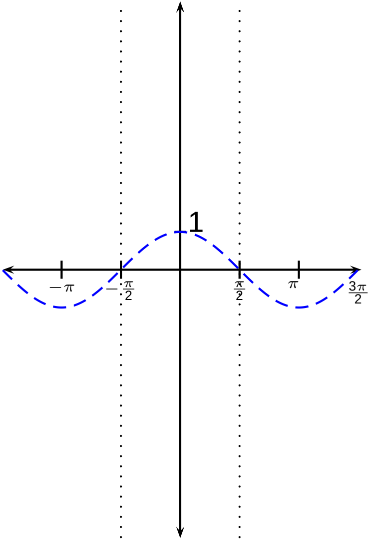
$y = \csc x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

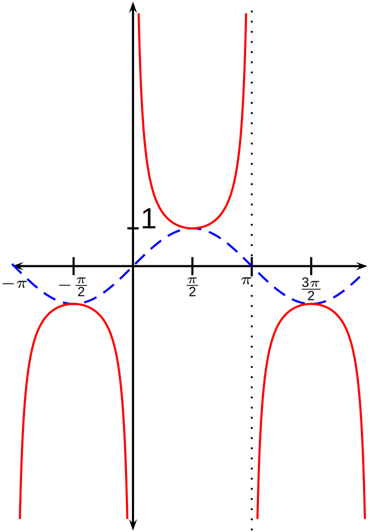
$y = \sec x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

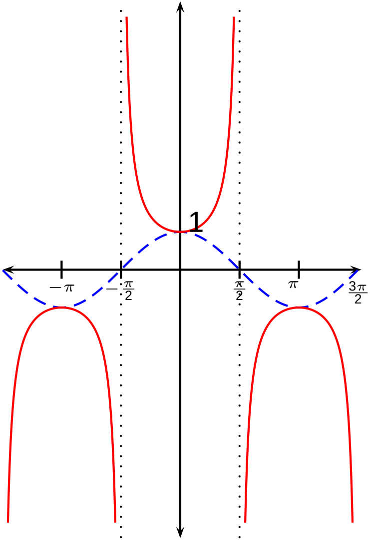
$y = \csc x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \csc x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$