Math 140

Lecture 4

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with modifications by T. Milev

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Outline

1. (1.4) The Tangent and Velocity Problems
    - The Tangent Problem

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2. (1.5) and (1.7) The Limit of a Function
   - One-sided Limits

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   - The Tangent Problem
2. (1.5) and (1.7) The Limit of a Function
   - One-sided Limits
3. (1.6) and (1.7) Calculating Limits Using Limit Laws

The Tangent Problem

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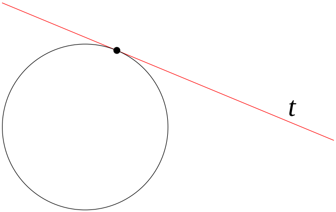

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- Moreover, a tangent should have the same "direction" as the curve at the point of contact.
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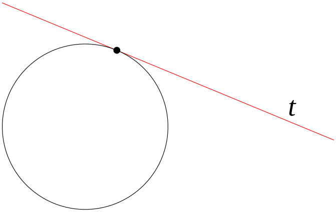

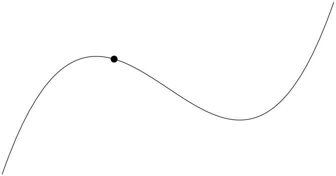

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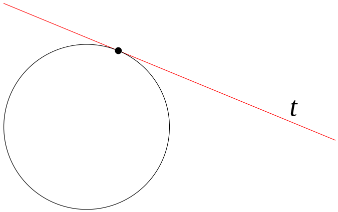

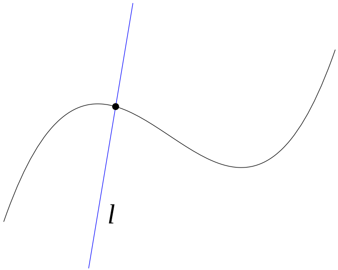

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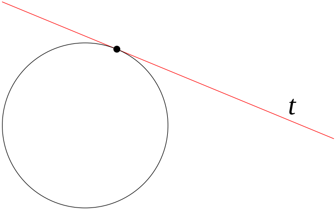

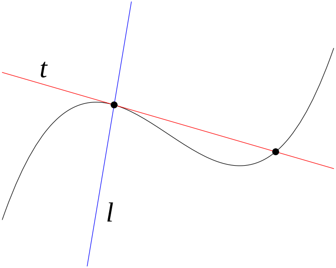

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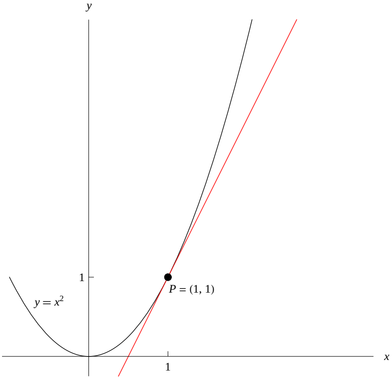
- The line *l* intersects the curve at exactly one point, but it doesn't look like a tangent.
- The line *t* does look like a tangent, but it intersects the curve at two points.



x	m_{PQ}	x	m_{PQ}
2		0	
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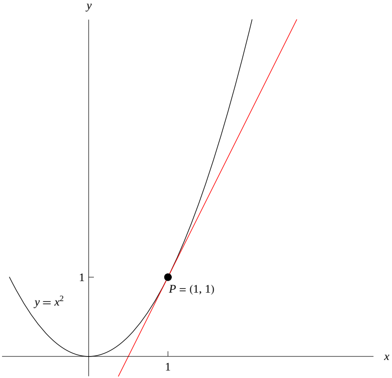

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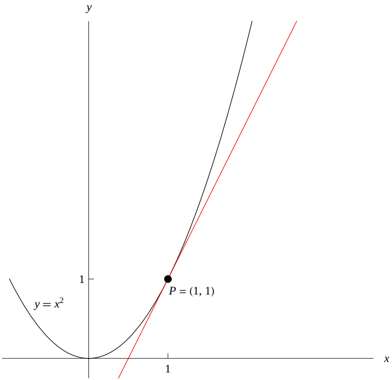

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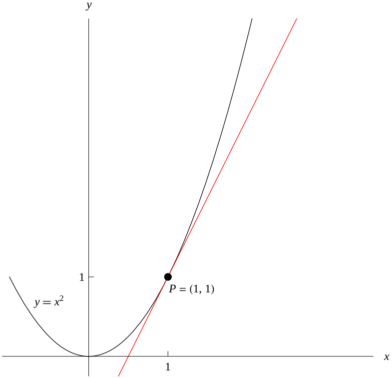

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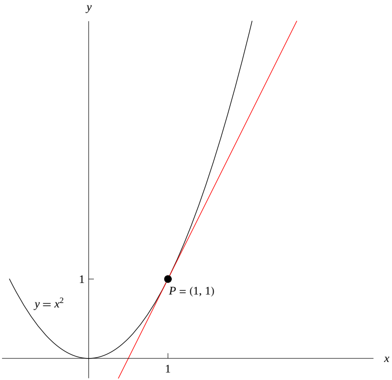

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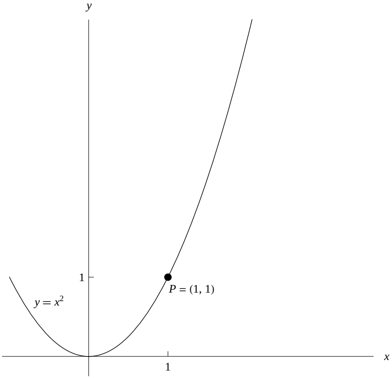

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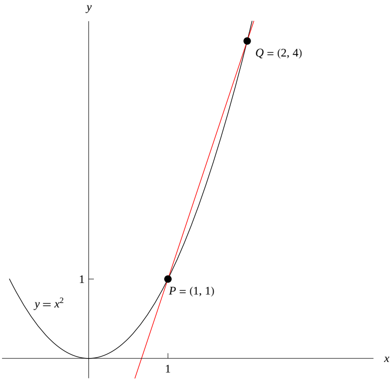

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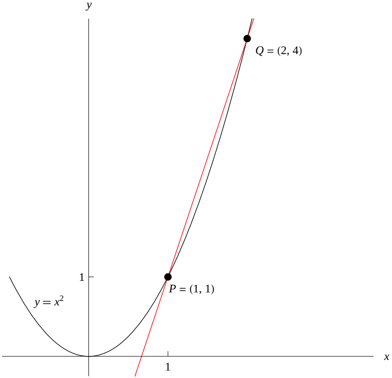

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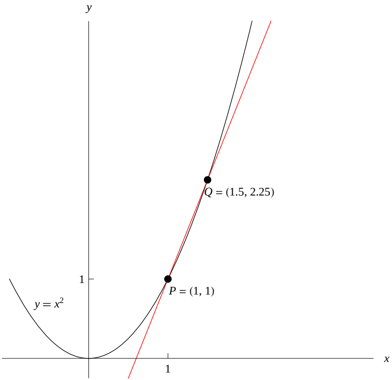

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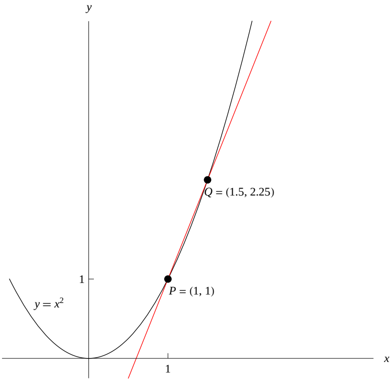

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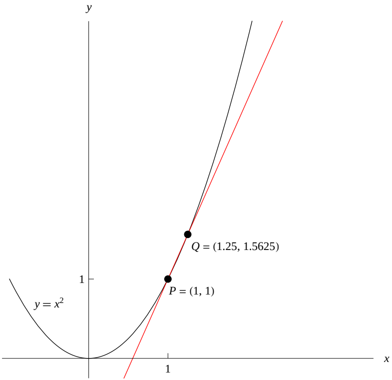

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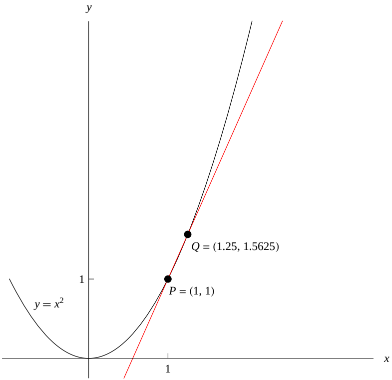

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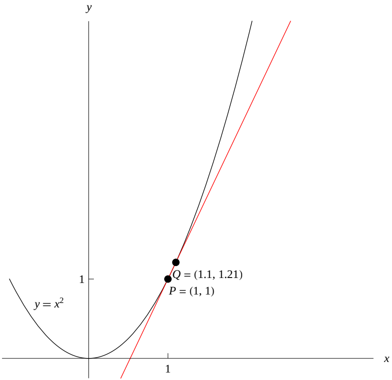

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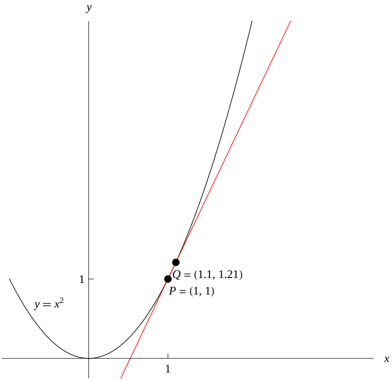

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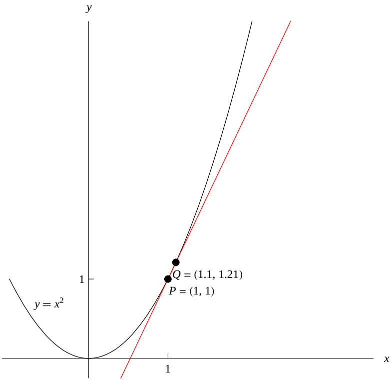

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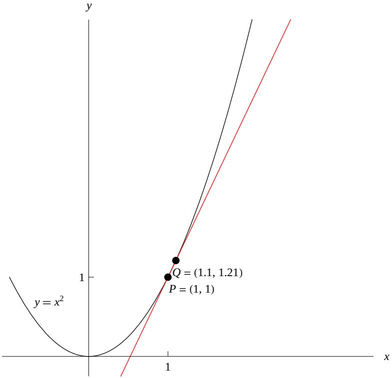

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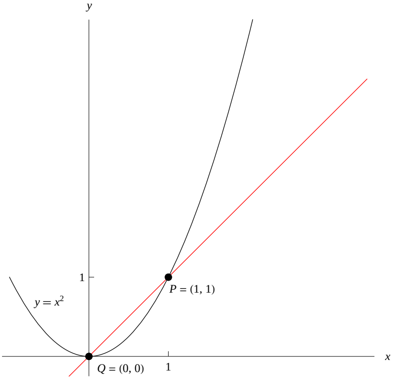

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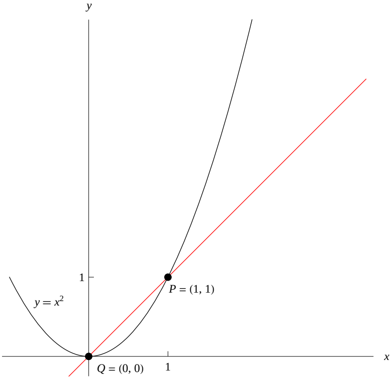

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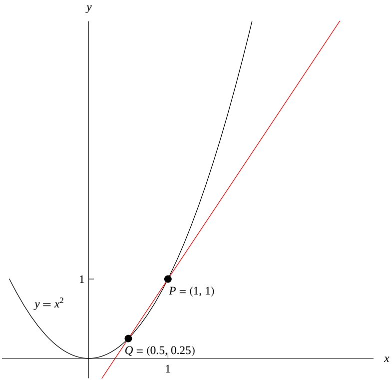

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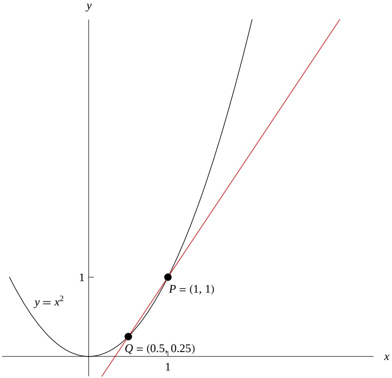

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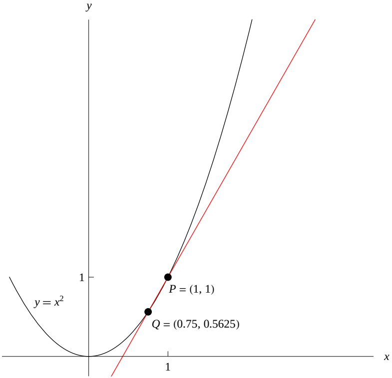

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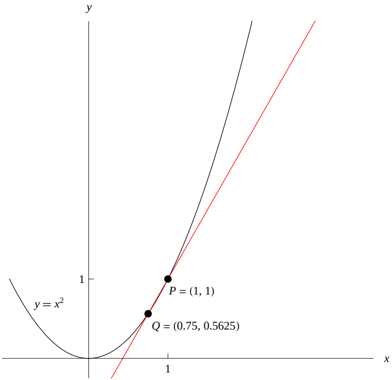

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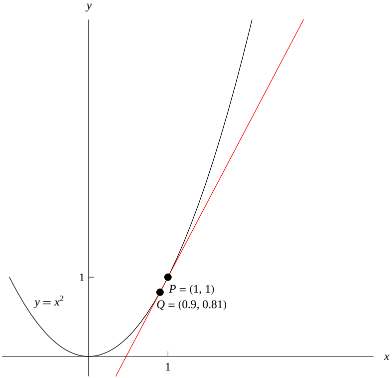

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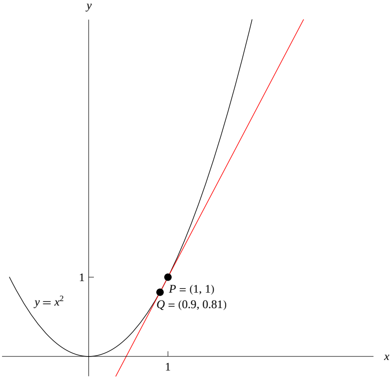

x	m_{PQ}	x	m_{PQ}
2	3	0	1
1.5	2.5	0.5	1.5
1.25	2.25	0.75	1.75
1.1	2.1	0.9	1.9
1.01	2.01	0.99	

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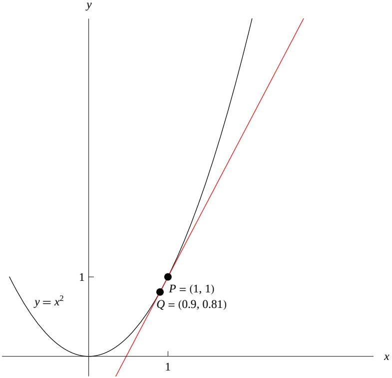

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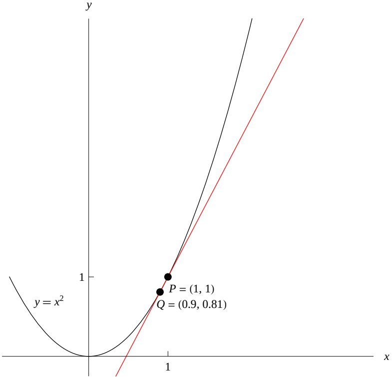

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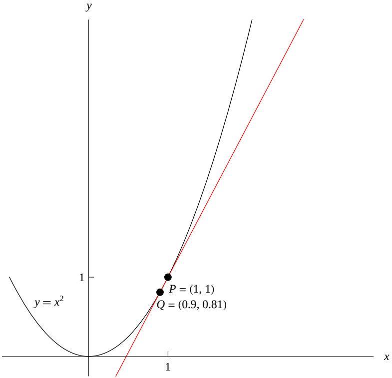

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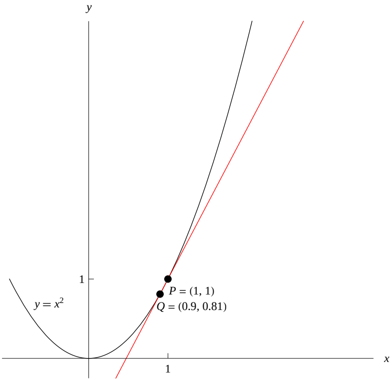
- Approximate solution: choose nearby point $Q = (x, x^2)$ on parabola. Let m_{PQ}= slope of line through P and Q.
- The closer x is to 1, the closer Q to P, the closer m_{PQ} is to 2.



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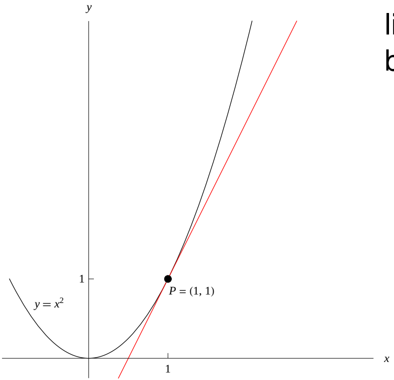
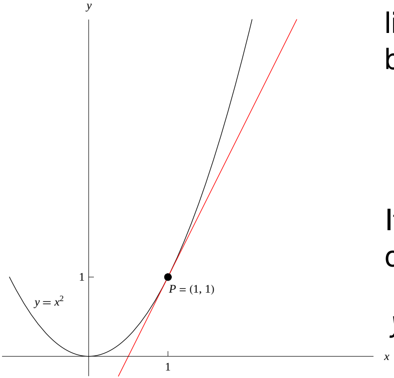
- The closer x is to 1, the closer Q to P, the closer m_{PQ} is to 2.
- This suggests the slope of the tangent should be 2.



We say that the slope of the tangent is the limit of the slope of the secants (limit will be defined later in the lecture). We write:

$$\lim_{Q \to P} m_{PQ} = m, \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$



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If the slope is indeed 2, then the equation of the tangent is

$$y - 1 = 2(x - 1), \qquad \text{or} \qquad y = 2x - 1.$$

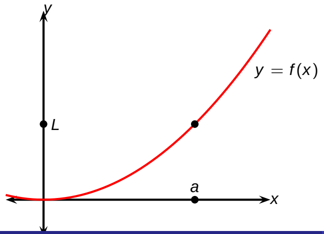
The Limit of a Function

Definition (The Limit of a Function)

We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.



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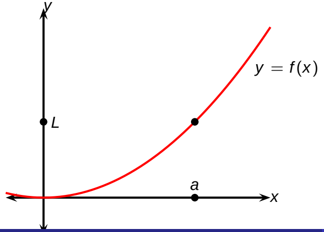
Equivalent formulation. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all x with $0 < |x - a| < \delta$



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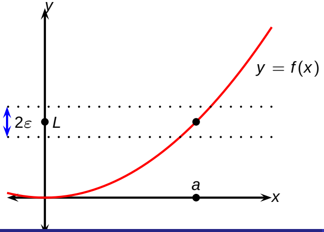

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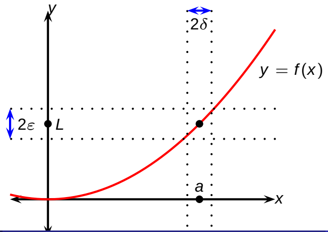

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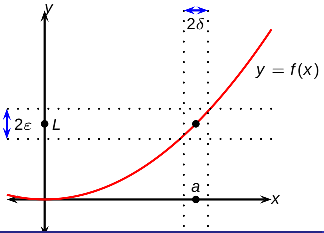

Example

- Guess the value of $\lim_{x\to 1} \frac{x-1}{x^2-1}$.

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x	$f(x)$	x	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

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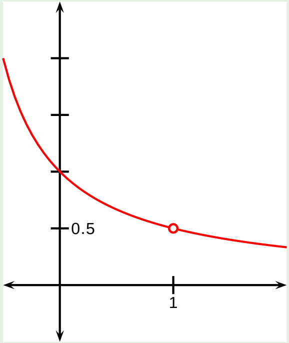

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x	$f(x)$	x	$f(x)$
± 1.0	0.841471	± 0.1	0.998334
± 0.5	0.958851	± 0.05	0.999583
± 0.4	0.973546	± 0.01	0.999983
± 0.3	0.985067	± 0.005	0.999995
± 0.2	0.993347	± 0.001	0.999999

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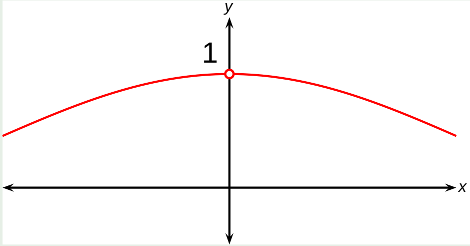

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- Notice that $\sin \frac{\pi}{x}$ is not defined at 0.
- It is defined at values near 0.

x	$f(x)$	x	$f(x)$
1	$\sin \pi = 0$	$\frac{1}{2}$	$\sin 2\pi = 0$
$\frac{1}{3}$	$\sin 3\pi = 0$	$\frac{1}{4}$	$\sin 4\pi = 0$
0.1	$\sin 10\pi = 0$	0.01	$\sin 100\pi = 0$

Example

- Guess the value of $\lim\limits_{x\to 0} \sin\frac{\pi}{x}$.
- Notice that $\sin\frac{\pi}{x}$ is not defined at 0.
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- We could guess that the limit is 0.

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Example

- Guess the value of $\lim_{x \to 0} \sin \frac{\pi}{x}$.
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- We could guess that the limit is 0.
- In this case, the guess is wrong.

x	$f(x)$	x	$f(x)$
1	$\sin \pi = 0$	$\frac{1}{2}$	$\sin 2\pi = 0$
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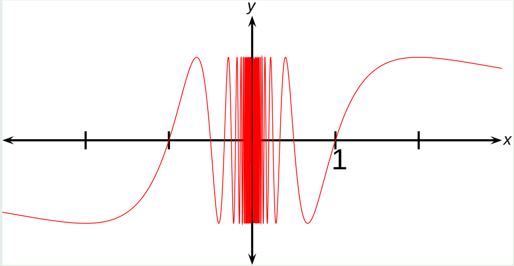
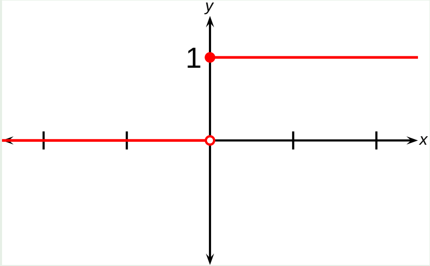


One-sided Limits

Example

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$



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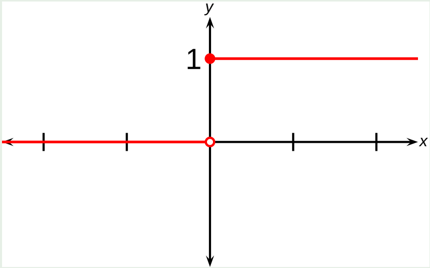

- As t approaches 0 from the left, $H(t)$ approaches 0.

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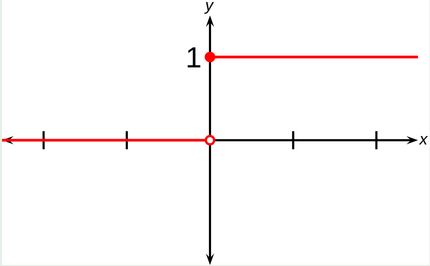

- As t approaches 0 from the left, $H(t)$ approaches 0.
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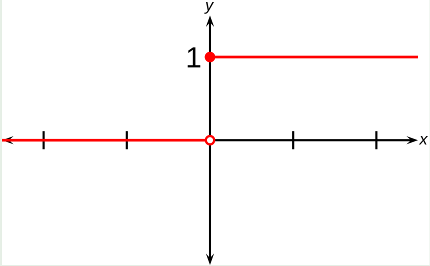

- As t approaches 0 from the left, $H(t)$ approaches 0.
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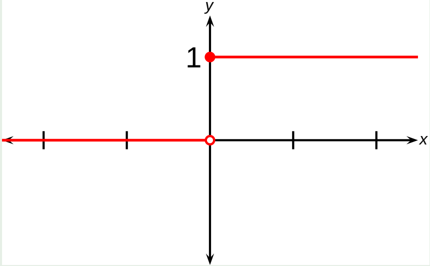


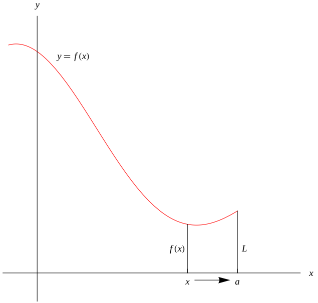
- As t approaches 0 from the left, $H(t)$ approaches 0.
- As t approaches 0 from the right, $H(t)$ approaches 1.
- There is no single number that $H(t)$ approaches as t approaches 0.
- Therefore $\lim\limits_{t \to 0} H(t)$ doesn't exist.

Definition (Left-hand Limit)

We write

$$\lim_{x \to a^-} f(x) = L \qquad \text{or} \qquad \lim_{\substack{x \to a \\ x < a}} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and less than a.



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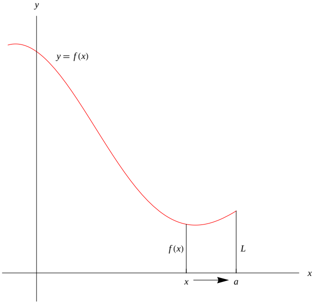


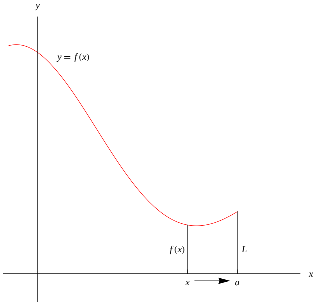
We can define a right-hand limit similarly.

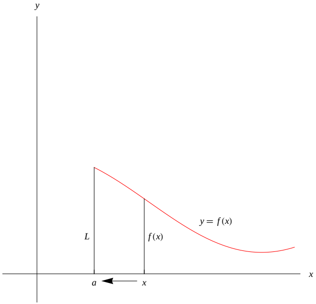
Definition (Right-hand Limit)

We write

$$\lim_{x \to a^+} f(x) = L \qquad \text{or} \qquad \lim_{\substack{x \to a \\ x > a}} f(x) = L$$

and say the right-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and greater than a.





We can define a right-hand limit similarly.

By comparing the definitions, we can see that

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$\lim_{x \to 1^-} g(x) =$ $\qquad$ $\lim_{x \to 3^-} g(x) =$

$\lim_{x \to 1^+} g(x) =$ $\qquad$ $\lim_{x \to 3^+} g(x) =$

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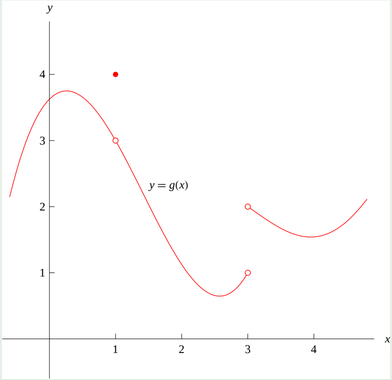

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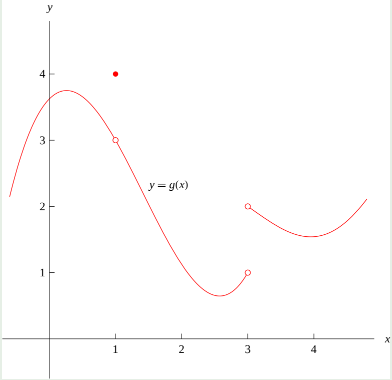

By comparing the definitions, we can see that

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\lim_{x\to 1^-} g(x) = 3 \qquad \lim_{x\to 3^-} g(x) =$$

$$\lim_{x\to 1^+} g(x) = \qquad \lim_{x\to 3^+} g(x) =$$

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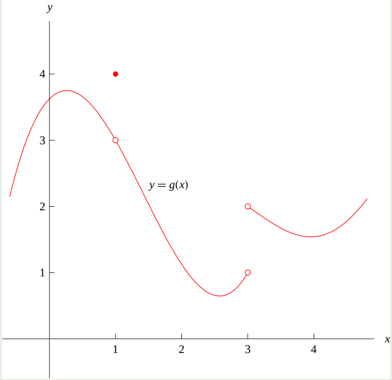

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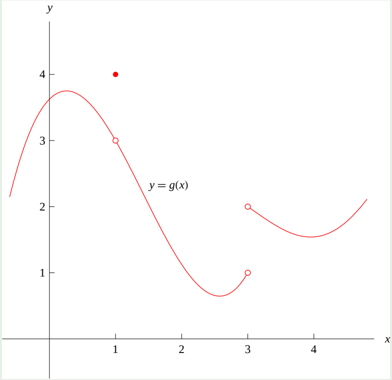

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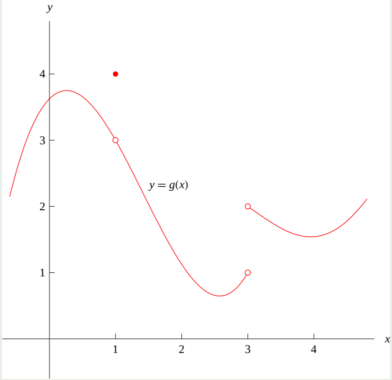

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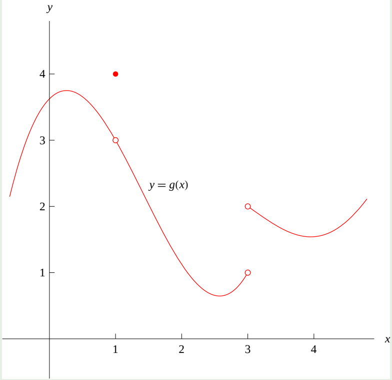

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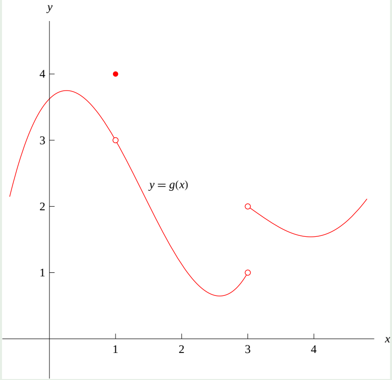

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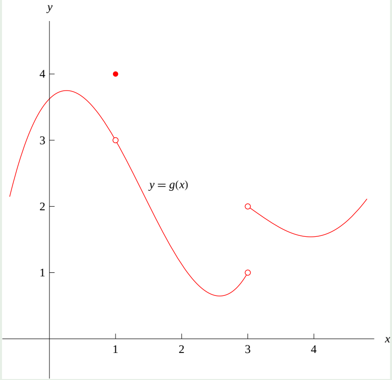

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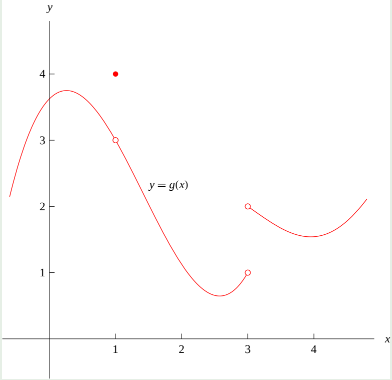
Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\lim_{x\to 1^-} g(x) = 3 \qquad \lim_{x\to 3^-} g(x) = 1$$

$$\lim_{x\to 1^+} g(x) = 3 \qquad \lim_{x\to 3^+} g(x) =$$

$$\lim_{x\to 1} g(x) = 3 \qquad \lim_{x\to 3} g(x) =$$



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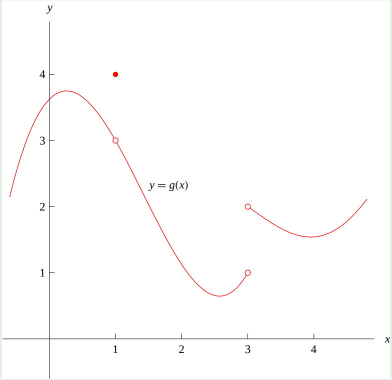

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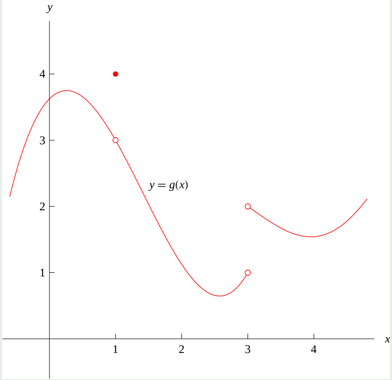
$$\lim_{x\to 1} g(x) = 3 \qquad \lim_{x\to 3} g(x) =$$



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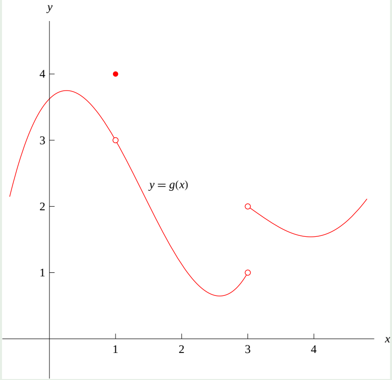

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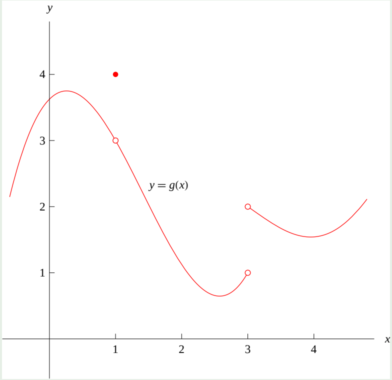
$$\lim_{x \to 1} g(x) = 3 \qquad \lim_{x \to 3} g(x) = \text{DNE}$$



Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist* ($\pm\infty$ ***not allowed***). *Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

Calculating Limits Using Limit Laws

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Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist* ($\pm\infty$ ***not allowed***). *Then*

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$.
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

Sum Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

*Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist ($\pm\infty$ **not allowed**). Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
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Difference Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist* ($\pm\infty$ ***not allowed***). *Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
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Constant Multiple Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

*Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist ($\pm\infty$ **not allowed**). Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

Product Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist* ($\pm\infty$ ***not allowed***). *Then*

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$
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4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0.$

Quotient Law

Here are some other useful limit laws:

6. $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$
7. $\lim\limits_{x\to a} c = c$.
8. $\lim\limits_{x\to a} x = a$.
9. $\lim\limits_{x\to a} x^n = a^n$.
10. $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, if $a > 0$.
11. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, if $\lim_{x\to a} f(x) > 0$.

Here are some other useful limit laws:

6. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$
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Power Law

Here are some other useful limit laws:

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7. $\lim\limits_{x \to a} c = c.$
8. $\lim\limits_{x \to a} x = a.$
9. $\lim\limits_{x \to a} x^n = a^n.$
10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, if $a > 0$.
11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, if $\lim_{x \to a} f(x) > 0$.

Root Law

Here are some other useful limit laws:

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$
7. $\lim_{x\to a} c = c$.
8. $\lim_{x\to a} x = a$.
9. $\lim_{x\to a} x^n = a^n$.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, if $a > 0$.
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if $\lim_{x\to a} f(x) > 0$.

Direct Substitution

Example

Evaluate the limit and justify each step:

$$\lim_{x \to 5} (2x^2 - 3x + 4)$$

Example

Evaluate the limit and justify each step:

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 \qquad \text{Law}$$

Example

Evaluate the limit and justify each step:

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 \qquad \text{Law 1}$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
& \lim_{x\to 5}(2x^2 - 3x + 4) & \\
= {} & \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 & \text{Law 1} \\
= {} & \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 & \text{Law}
\end{aligned}
$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned}
&\lim_{x\to 5}(2x^2 - 3x + 4) \\
=\ &\lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 && \text{Law 1} \\
=\ &\lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{Law 2}
\end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 \qquad \text{Law 1}$$

$$= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 \qquad \text{Law 2}$$

$$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 \qquad \text{Law}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned}
&\lim_{x\to 5}(2x^2 - 3x + 4) & \\
=\ &\lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 & \text{Law 1} \\
=\ &\lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 & \text{Law 2} \\
=\ &2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 & \text{Law 3}
\end{aligned}$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
&\lim_{x\to 5}(2x^2 - 3x + 4) \\
&= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 && \text{Law 1} \\
&= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{Law 2} \\
&= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Law 3} \\
&= 2\cdot 5^2 - 3\cdot 5 + 4 && \text{Laws}
\end{aligned}
$$

Example

Evaluate the limit and justify each step:

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

$$= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 \qquad \text{Law 1}$$

$$= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 \qquad \text{Law 2}$$

$$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 \qquad \text{Law 3}$$

$$= 2\cdot 5^2 - 3\cdot 5 + 4 \qquad \text{Laws 7}$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
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&= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 & \text{Law 1} \\
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&= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 & \text{Law 3} \\
&= 2\cdot 5^2 - 3\cdot 5 + 4 & \text{Laws 7}
\end{aligned}
$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
&\lim_{x\to 5}(2x^2-3x+4) && \\
&= \lim_{x\to 5}(2x^2-3x)+\lim_{x\to 5}4 && \text{Law 1}\\
&= \lim_{x\to 5}(2x^2)-\lim_{x\to 5}(3x)+\lim_{x\to 5}4 && \text{Law 2}\\
&= 2\lim_{x\to 5}x^2-3\lim_{x\to 5}x+\lim_{x\to 5}4 && \text{Law 3}\\
&= 2\cdot 5^2-3\cdot 5+4 && \text{Laws 7, 8}
\end{aligned}
$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned}
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&= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 && \text{Law 1} \\
&= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{Law 2} \\
&= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Law 3} \\
&= 2\cdot 5^2 - 3\cdot 5 + 4 && \text{Laws 7, 8}
\end{aligned}$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
&\lim_{x\to 5}(2x^2 - 3x + 4) && \\
&= \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 && \text{Law 1} \\
&= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{Law 2} \\
&= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Law 3} \\
&= 2\cdot 5^2 - 3\cdot 5 + 4 && \text{Laws 7, 8, and 9}
\end{aligned}
$$

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
& \lim_{x\to 5}(2x^2 - 3x + 4) & \\
= & \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 & \text{Law 1} \\
= & \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 & \text{Law 2} \\
= & \, 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 & \text{Law 3} \\
= & \, 2\cdot 5^2 - 3\cdot 5 + 4 & \text{Laws 7, 8, and 9} \\
= & \, 39. &
\end{aligned}
$$