Math 140

Lecture 5

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with modifications by T. Milev

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Outline

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Example (Limit Laws)

Evaluate the limit and justify each step:

$$\lim_{x \to 3} \frac{x+2}{\sqrt{x-1}(x+1)^2}$$

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$$= \frac{3+2}{\sqrt{3-1}\,(3+1)^2} = \frac{5}{16\sqrt{2}}. \qquad \text{Laws 8 and 7}$$

Theorem (Direct Substitution)

Let f be an algebraic function. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim_{x \to a} f(x) = f(a)$.

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This theorem is a partial case of the following theorem.

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Continuous functions will be defined later in this lecture.

Example (Limit with Direct Substitution)

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Plug in 3: $\dfrac{(3)+2}{\sqrt{(3)-1}((3)+1)^2} = \underline{\qquad}$

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Therefore $\lim_{x\to 3} \dfrac{x+2}{\sqrt{x-1}(x+1)^2} = \dfrac{5}{16\sqrt{2}}.$

Example (Limit in Which Direct Substitution Doesn't Work)

$$\text{Find} \quad \lim_{x \to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$$

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Zero over zero is undefined, so we can't use direct substitution.

When computing a limit as x approaches a, we don't care what happens when $x = a$. This gives the following useful fact:

$$\text{If} \quad f(x) = g(x)$$

when $x \neq a$,

$$\text{then} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x),$$

provided the limit exists.

We can use this fact to find $\lim_{x \to a} f(x)$ when $f(a)$ has the form $\frac{0}{0}$. In such a case, we use algebra to find a function $g(x)$ that agrees with $f(x)$ at all points except $x = a$. Here are some common techniques.

1. Factoring.
2. Using a conjugate radical.
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1. Factoring.
2. Using a conjugate radical.
3. Finding a common denominator.
4. Using Taylor/Maclaurin series expansion. Studied in Calc II.

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$$= \frac{10}{-1}$$

Example (Limit with Factoring)

$$\text{Find} \quad \lim_{x \to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$$

$$\text{Plug in 3:} \quad \frac{(3)^3 - 3(3)^2 + (3) - 3}{(3)^2 - 7(3) + 12} = \frac{0}{0}$$

Zero over zero is undefined, so we can't use direct substitution.

$$\text{Factor:} \quad \lim_{x \to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \lim_{x \to 3} \frac{(x^2 + 1)(x - 3)}{(x - 4)(x - 3)}$$

$$= \lim_{x \to 3} \frac{x^2 + 1}{x - 4}$$

$$\text{Plug in 3:} \quad \lim_{x \to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \frac{(3)^2 + 1}{(3) - 4}$$

$$= \frac{10}{-1}$$

$$= -10.$$

Example

Find $\displaystyle\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$

Example

Find $\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

Plug in 0: $\frac{\sqrt{(0)^2 + 9} - 3}{(0)^2} = -$

Example

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Find $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Plug in 0: $\displaystyle\frac{\sqrt{(0)^2+9}-3}{(0)^2} = \frac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.

Use a conjugate radical:

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{}{}$$

Example

Find $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Plug in 0: $\displaystyle\frac{\sqrt{(0)^2+9}-3}{(0)^2} = \frac{0}{0}$

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Example

Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

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$$= \lim_{t \to 0} \frac{}{t^2(\sqrt{t^2+9}+3)}$$

Example

Find $\displaystyle \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

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$$= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

Example

Find $\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

Plug in 0: $\frac{\sqrt{(0)^2 + 9} - 3}{(0)^2} = \frac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.

Use a conjugate radical:

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3}$$

$$= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} = \lim_{t \to 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)}$$

Example

Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Plug in 0: $\frac{\sqrt{(0)^2+9}-3}{(0)^2} = \frac{0}{0}$

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$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

Example

Find $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Plug in 0: $\displaystyle\frac{\sqrt{(0)^2+9}-3}{(0)^2} = \frac{0}{0}$

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Plug in 0: $\displaystyle = \frac{1}{\sqrt{(0)^2+9}+3}$

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$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

Plug in 0: $\displaystyle = \frac{1}{\sqrt{(0)^2+9}+3} = \frac{1}{6}.$

Example

Find $\lim_{x\to 1} g(x)$, where

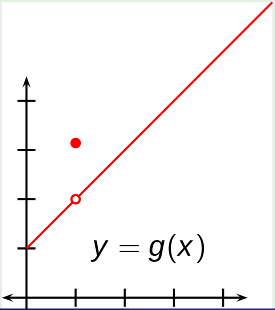
$$g(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ \pi & \text{if} \quad x = 1 \end{cases}$$

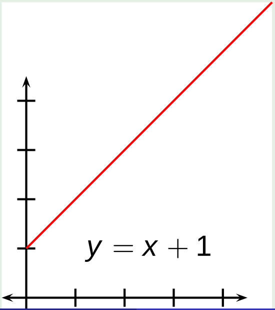
Example

Find $\lim_{x\to 1} g(x)$, where

$$g(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ \pi & \text{if} \quad x = 1 \end{cases}$$

g agrees with the function $f(x) = x + 1$ at every point except for $x = 1$.





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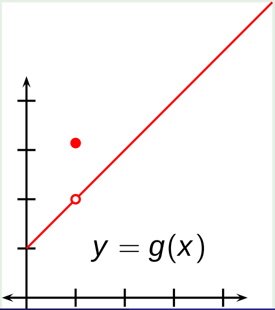
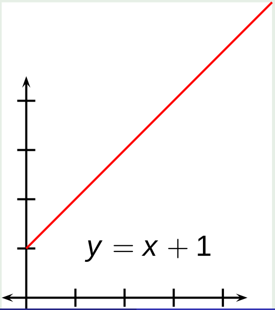
$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) =$$





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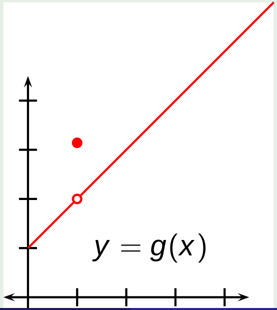
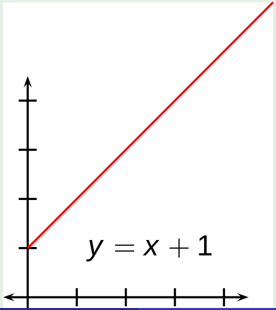
$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x+1) = 2.$$





Example (Limit with Factoring)

Find $\displaystyle\lim_{h\to 0}\frac{(3+h)^2-9}{h}$

Example (Limit with Factoring)

Find $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Plug in 0: $\displaystyle\frac{(3+(0))^2-9}{(0)} = -$

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Find $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

Plug in 0: $\frac{(3+(0))^2 - 9}{(0)} = \frac{0}{0}$

Example (Limit with Factoring)

Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Plug in 0: $\frac{(3+(0))^2-9}{(0)} = \frac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.

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Find $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

Plug in 0: $\frac{(3+(0))^2 - 9}{(0)} = \frac{0}{0}$

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$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

Example (Limit with Factoring)

Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

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$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h}$$

Example (Limit with Factoring)

Find $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$

Plug in 0: $\dfrac{(3+(0))^2 - 9}{(0)} = \dfrac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h}$$

Factor: $= \lim_{h \to 0} \dfrac{\quad}{h}$

Example (Limit with Factoring)

Find $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

Plug in 0: $\frac{(3+(0))^2 - 9}{(0)} = \frac{0}{0}$

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Factor: $$= \lim_{h \to 0} \frac{h(6+h)}{h}$$

Example (Limit with Factoring)

Find $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Plug in 0: $\displaystyle\frac{(3+(0))^2-9}{(0)} = \frac{0}{0}$

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$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h}$$

$$\text{Factor:} \quad = \lim_{h\to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h\to 0} (6+h)$$

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Factor: $= \lim_{h\to 0} \frac{h(6+h)}{h}$

$$= \lim_{h\to 0}(6+h)$$

Plug in 0: $= (6+(0))$

Example (Limit with Factoring)

Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Plug in 0: $\frac{(3+(0))^2-9}{(0)} = \frac{0}{0}$

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$$\text{Factor:} \quad = \lim_{h\to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h\to 0} (6+h)$$

$$\text{Plug in 0:} \quad = (6+(0)) = 6.$$

Recall from section 2.2:

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

We can use this to find the limit of a piecewise defined function, or show that it doesn't exist.

Example

If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

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If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

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$$\lim_{x\to 4^+} f(x)$$

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If

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determine whether $\lim_{x \to 4} f(x)$ exists.

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4}$$

Example

If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8-2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Example

If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8-2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x\to 4^-} f(x)$$

Example

If

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$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x)$$

Example

If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

Example

If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x) = 8 - 2\cdot 4 = 0$$

The left and right hand limits are equal. Therefore the limit exists and

$$\lim_{x\to 4} f(x) = 0.$$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Theorem (The Squeeze Theorem)

Suppose $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then

$$\lim_{x \to a} g(x) = L.$$

Example

Show that $\lim_{x \to 0} x^2 \sin \frac{8}{x} = 0$.

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WRONG: $$\lim_{x\to 0} x^2 \sin \frac{8}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin \frac{8}{x}$$

Doesn't work because $\lim_{x\to 0} \sin \frac{8}{x}$ doesn't exist.

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Doesn't work because $\lim_{x\to 0} \sin\frac{8}{x}$ doesn't exist.

$$-1 \quad \leq \quad \sin\tfrac{8}{x} \quad \leq \quad 1.$$

Example

Show that $\lim_{x \to 0} x^2 \sin \frac{8}{x} = 0$.

WRONG: $$\lim_{x \to 0} x^2 \sin \frac{8}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{8}{x}$$

Doesn't work because $\lim_{x \to 0} \sin \frac{8}{x}$ doesn't exist.

$$\begin{array}{rcccl} -1 & \leq & \sin \frac{8}{x} & \leq & 1. \\ -x^2 & \leq & x^2 \sin \frac{8}{x} & \leq & x^2. \end{array}$$

Example

Show that $\lim_{x \to 0} x^2 \sin \frac{8}{x} = 0$.

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$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} (-x^2) = 0.$$

Example

Show that $\lim_{x\to 0} x^2 \sin\frac{8}{x} = 0$.

WRONG: $$\lim_{x\to 0} x^2 \sin\frac{8}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{8}{x}$$

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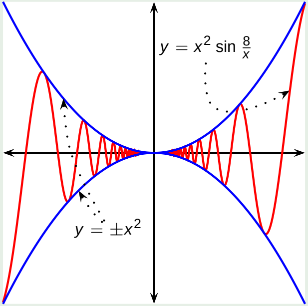

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$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0}(-x^2) = 0.$$

Therefore by the Squeeze Theorem

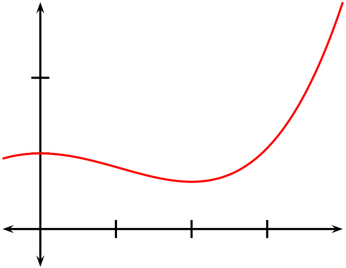
$$\lim_{x\to 0} x^2 \sin\frac{8}{x} = 0.$$

Continuity

Definition (Continuous at a Number)

A function *f* is continuous at a number *a* if

$$\lim_{x \to a} f(x) = f(a).$$



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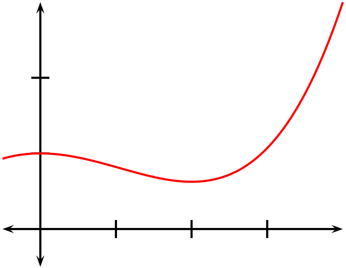

The definition (implicitly) requires the following.

1. $f(a)$ is defined (i.e., a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.

Definition (Discontinuous at a Number)

Suppose f is defined near a. We say f is discontinuous at a if it is not continuous at a.

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Discontinuous phenomena examples:

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- Motion of a vehicle with respect to time without sudden brakes.
- Orbits of planets and celestial bodies with respect to time.
- A person's height with respect to time.
- And many more.

Discontinuous phenomena examples:

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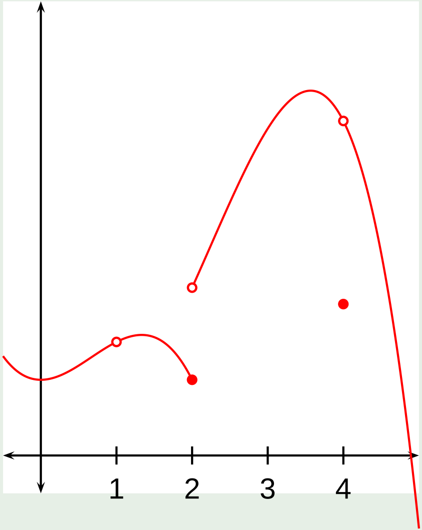
- A person's height with respect to time.
- And many more.

Discontinuous phenomena examples:

- Particle velocities during collisions and explosions.
- Electric current phenomena, gating events in porins (the event of a molecule passing in and out of a cell).
- Particle physics phenomena.
- And many more.

Example

The picture below shows a graph of a function f. At which numbers is f either discontinuous or not defined? Why?



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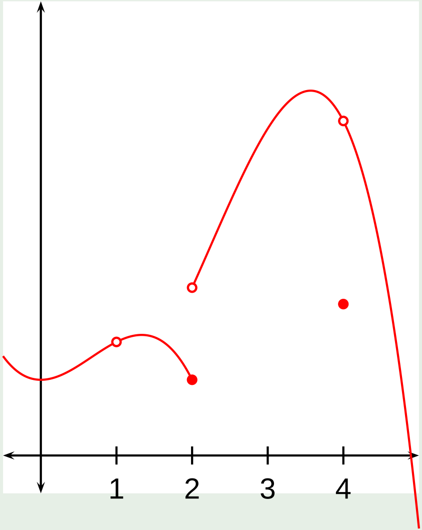

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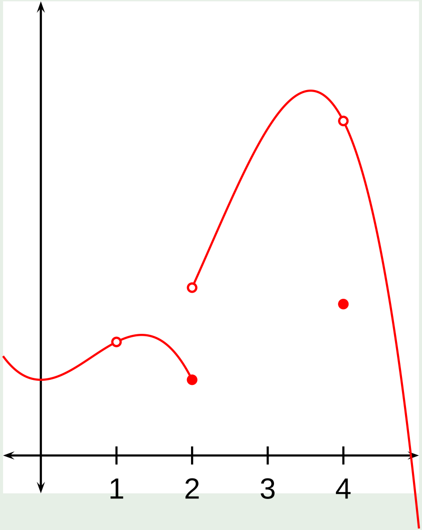

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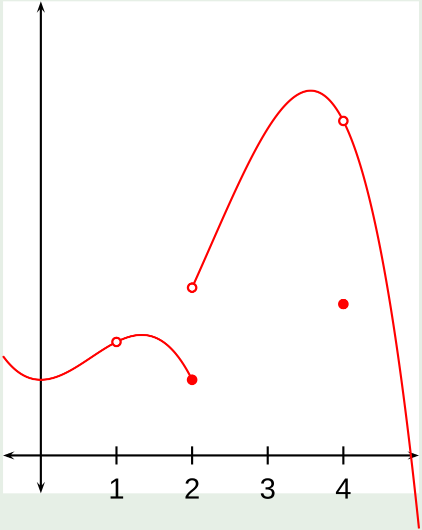

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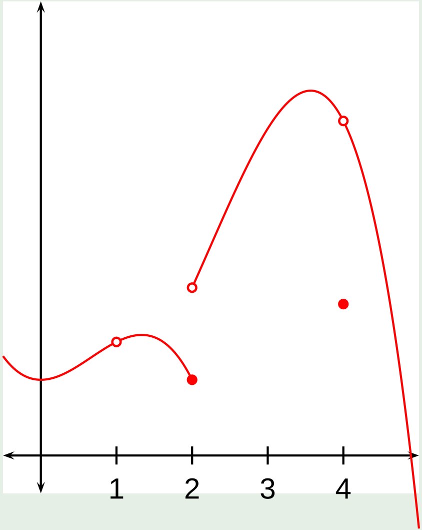

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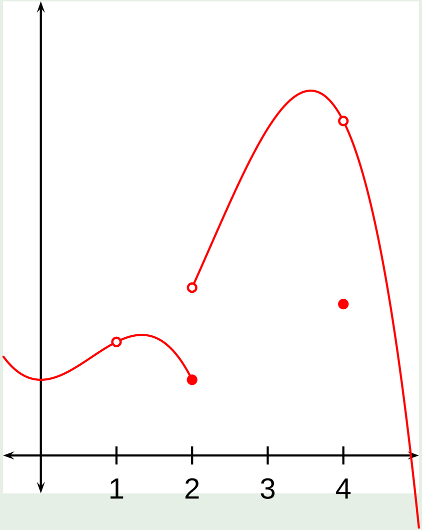

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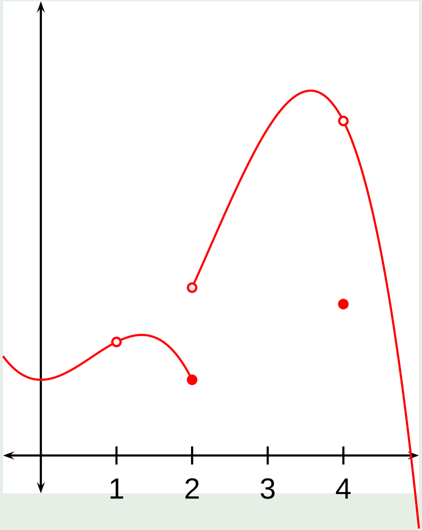

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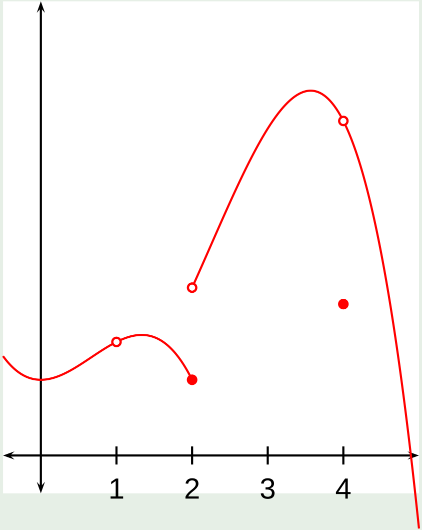

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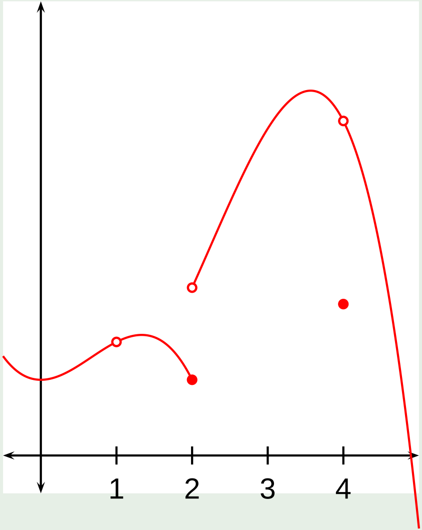

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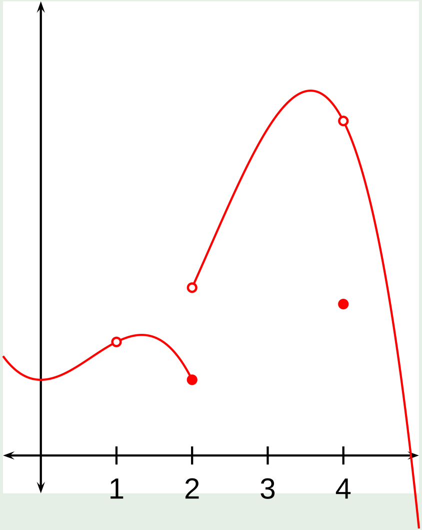

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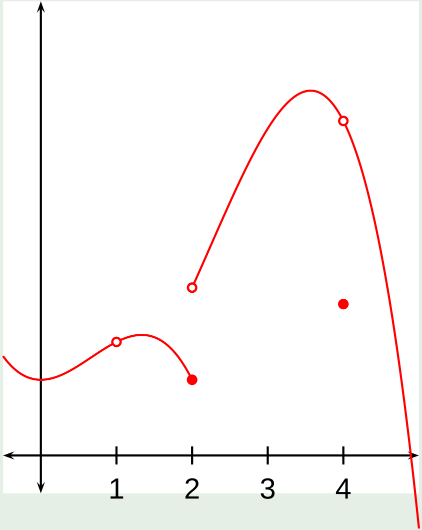

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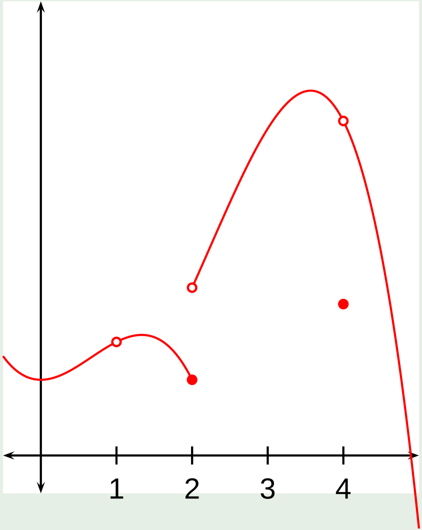

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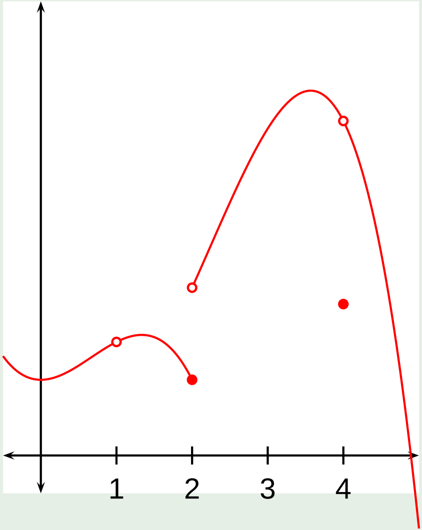

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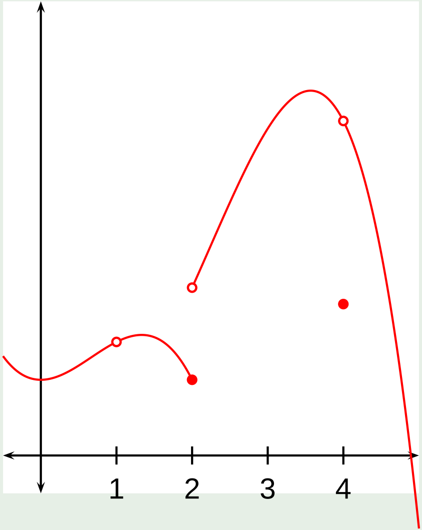

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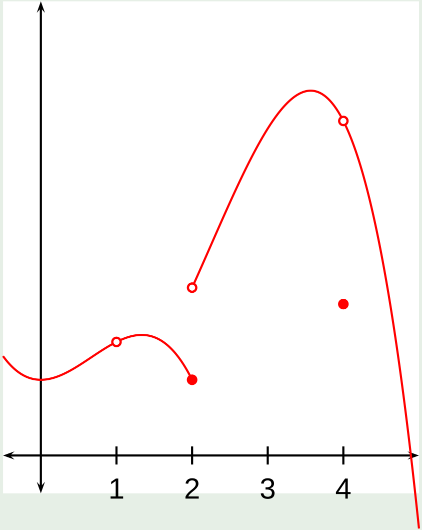

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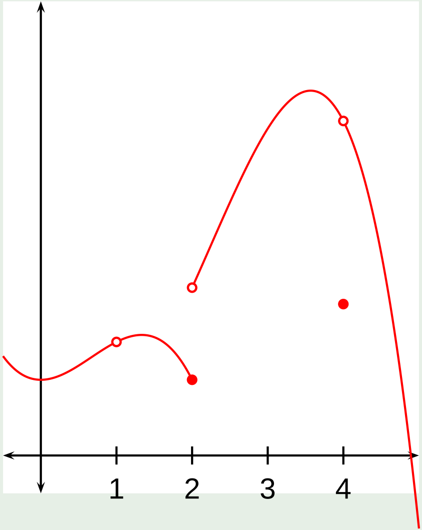

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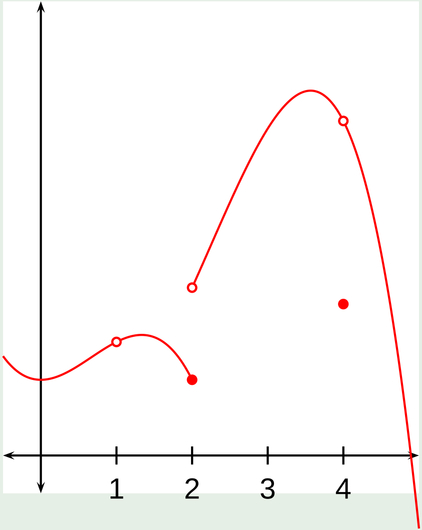

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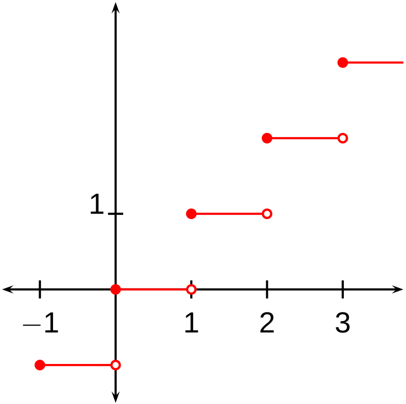
The greatest integer function $\lfloor x \rfloor$ is defined as the largest integer that is less than or equal to x.



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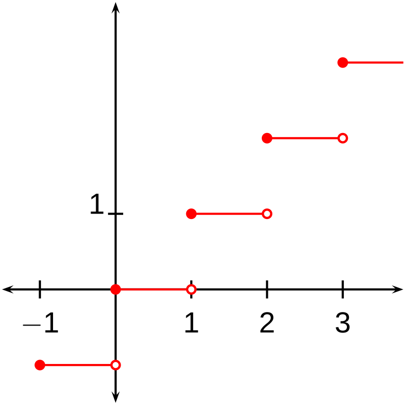

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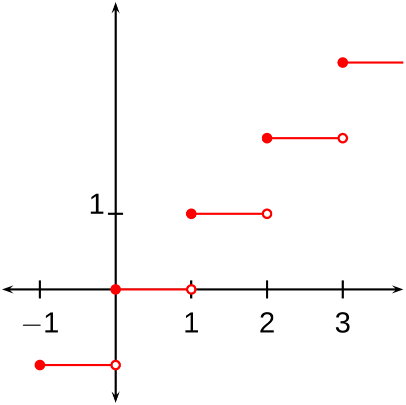

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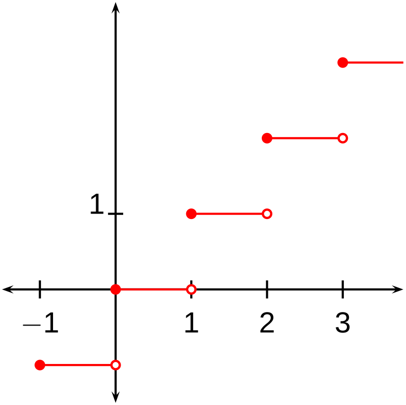

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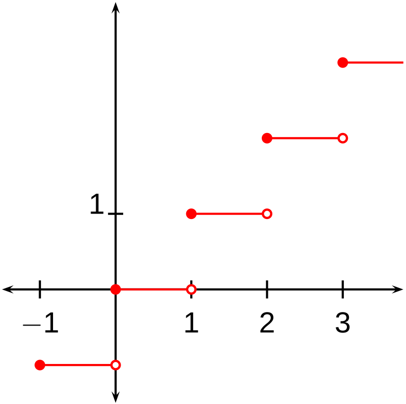

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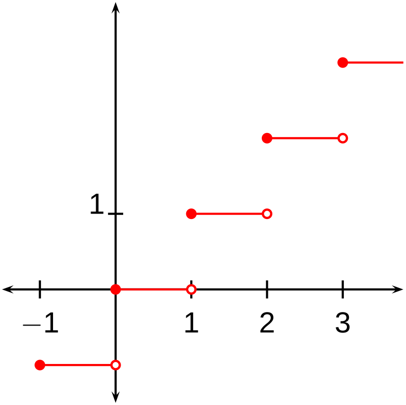

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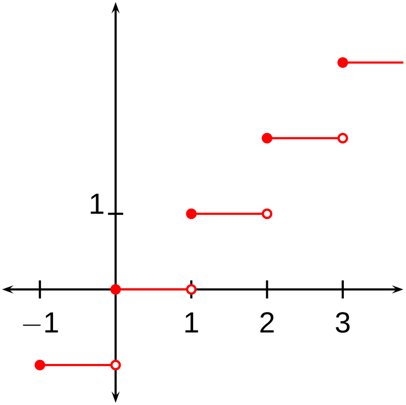

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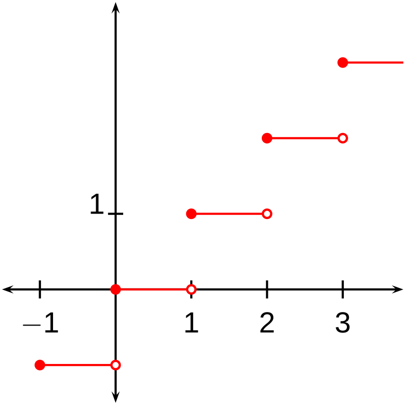

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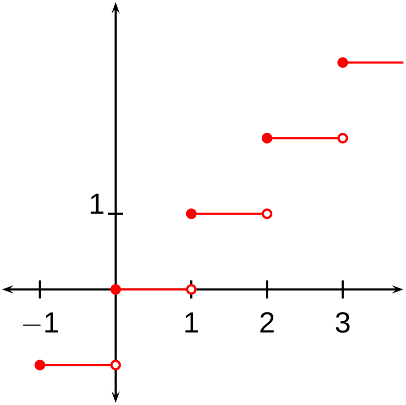

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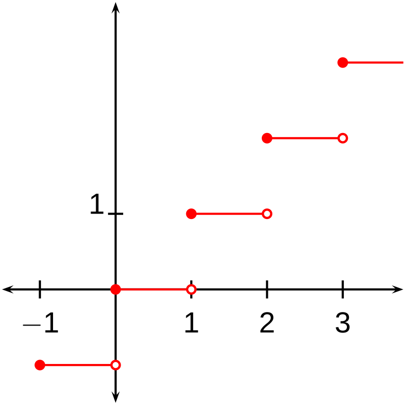

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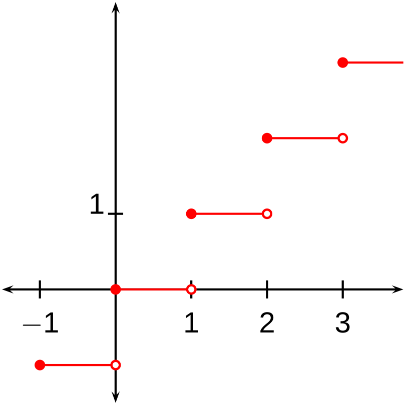

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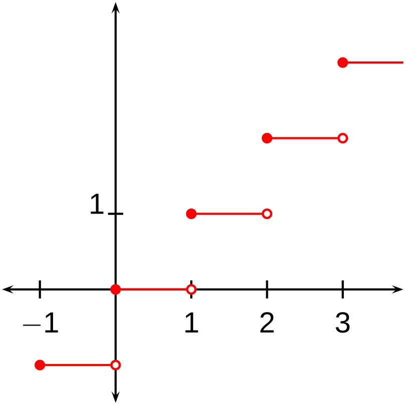

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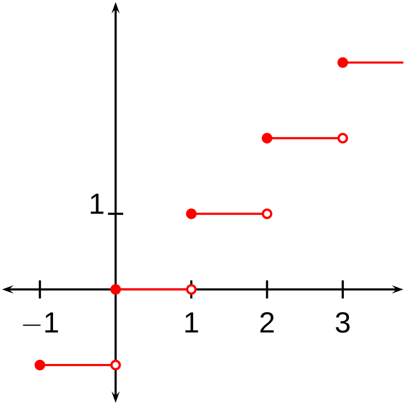

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Where is this function discontinuous?

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

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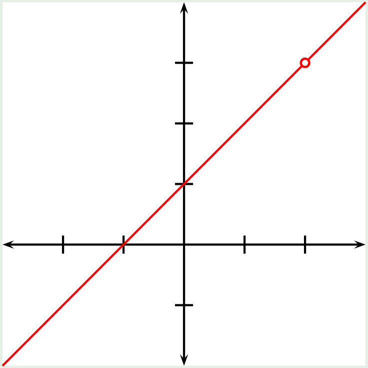

- $f(2)$ is not defined.
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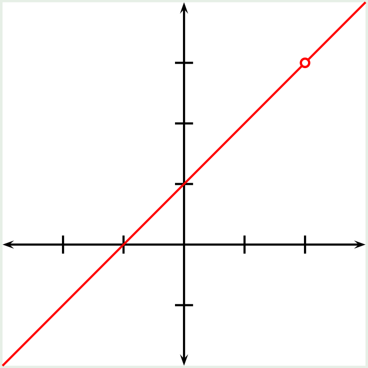
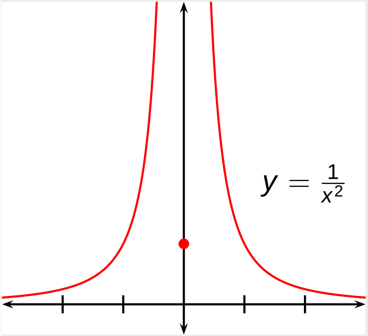


- $f(2)$ is not defined.
- Discontinuous at 2.
- This is called a removable discontinuity because we could remove it by redefining f at the single number 2.

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$



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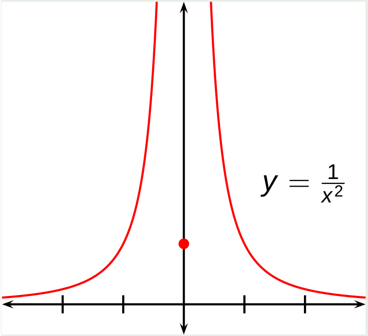

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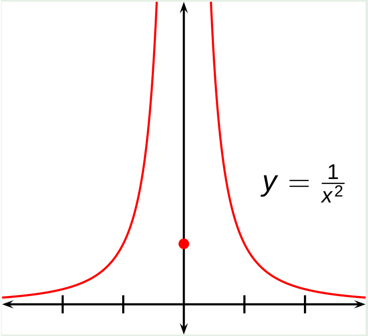

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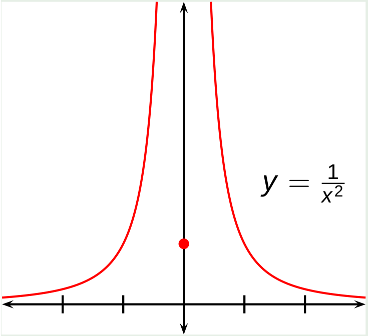

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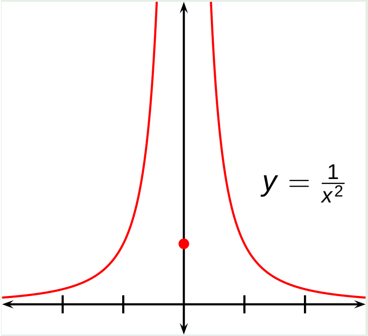

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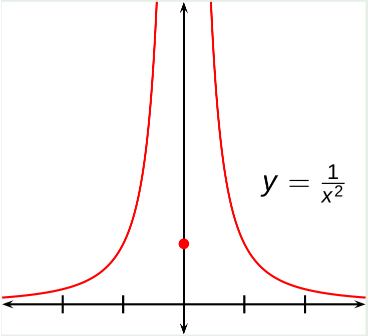
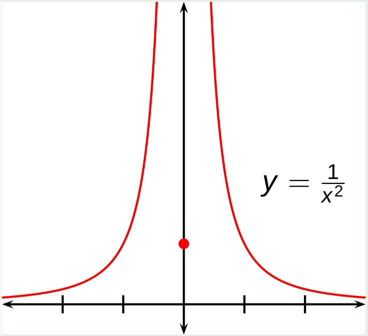


- $f(0)$ is defined ($f(0) = 1$).
- $\lim_{x \to 0} f(x)$ doesn't exist (∞).

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

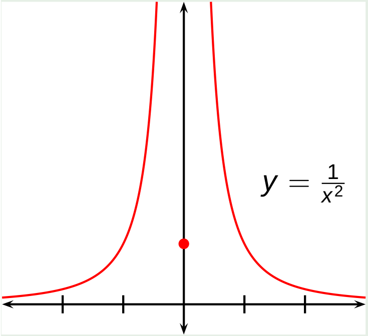


- $f(0)$ is defined ($f(0) = 1$).
- $\lim_{x \to 0} f(x)$ doesn't exist (∞).
- Discontinuous at 0.

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

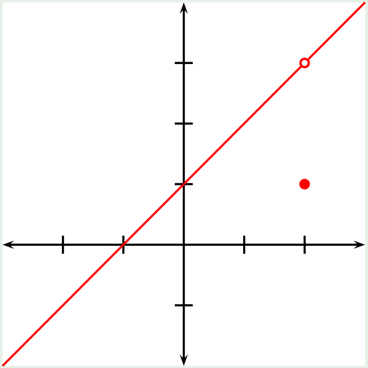


- $f(0)$ is defined ($f(0) = 1$).
- $\lim_{x \to 0} f(x)$ doesn't exist (∞).
- Discontinuous at 0.
- This is called an infinite discontinuity.

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$$



Example

Where is this function discontinuous?

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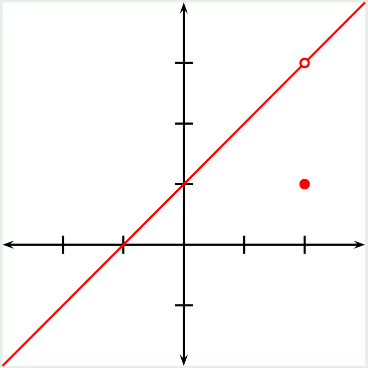

- $f(2)$
- $\lim_{x \to 2} f(x)$

Example

Where is this function discontinuous?

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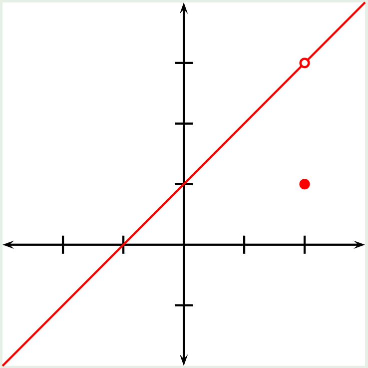

- $f(2)$
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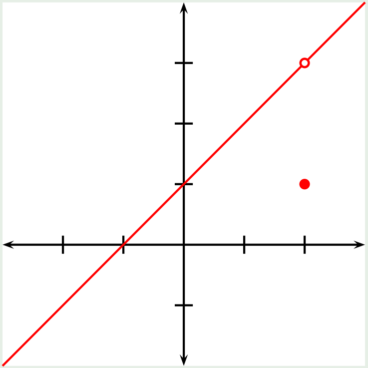

- $f(2)$ is defined ($f(2) = 1$).
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Example

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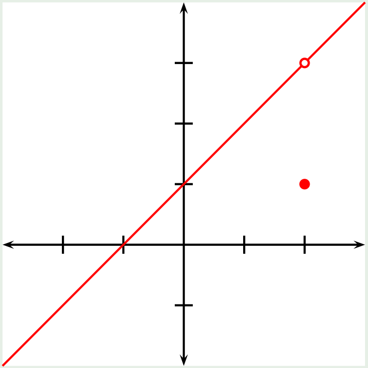

- $f(2)$ is defined ($f(2) = 1$).
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Example

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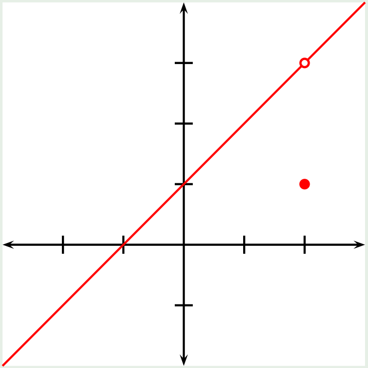
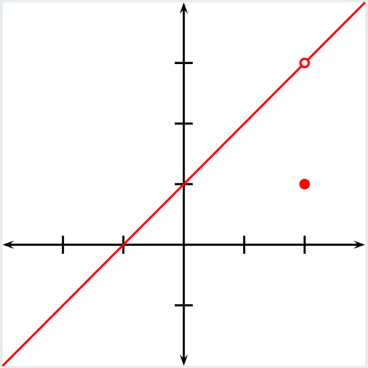


- $f(2)$ is defined ($f(2) = 1$).
- $\lim_{x \to 2} f(x)$ exists ($= 3$).

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$$



- $f(2)$ is defined ($f(2) = 1$).
- $\lim_{x\to 2} f(x)$ exists ($= 3$).
- $\lim_{x\to 2} f(x) \neq f(2)$.

Example

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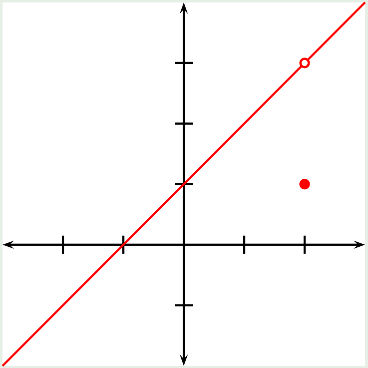

- $f(2)$ is defined ($f(2) = 1$).
- $\lim_{x\to 2} f(x)$ exists ($= 3$).
- $\lim_{x\to 2} f(x) \neq f(2)$.
- Discontinuous at 2.

Example

Where is this function discontinuous?

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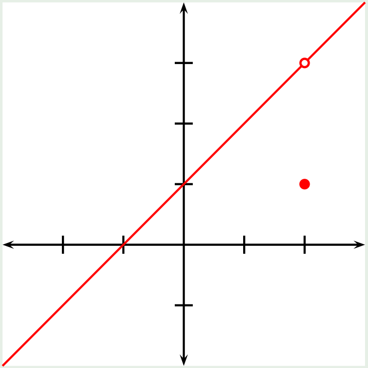
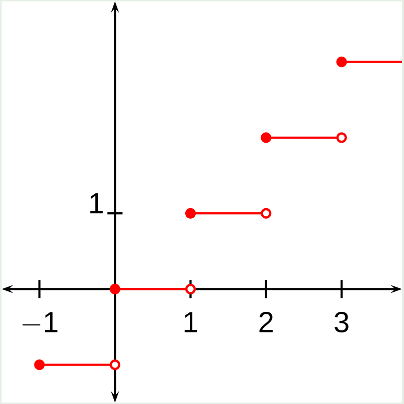


- $f(2)$ is defined ($f(2) = 1$).
- $\lim_{x \to 2} f(x)$ exists ($= 3$).
- $\lim_{x \to 2} f(x) \neq f(2)$.
- Discontinuous at 2.
- This is also called a removable discontinuity.

Example

Where is this function discontinuous?

$$f(x) = \lfloor x \rfloor$$



Example

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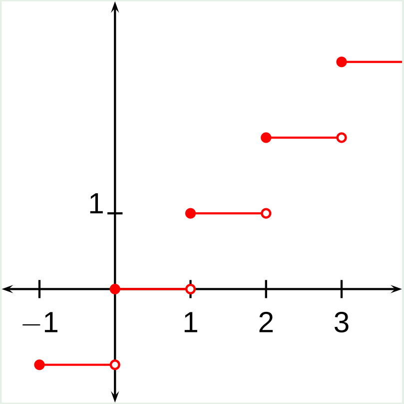

- $f(1)$
- $\lim_{x \to 1^+} f(x)$
- $\lim_{x \to 1^-} f(x)$
- $\lim_{x \to 1} f(x)$

Example

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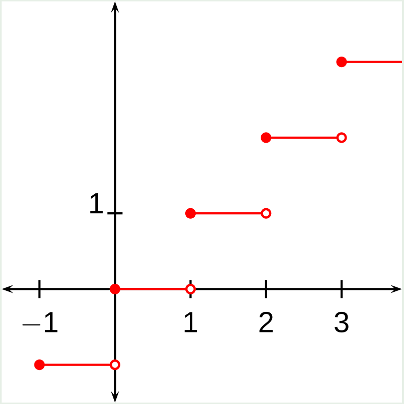

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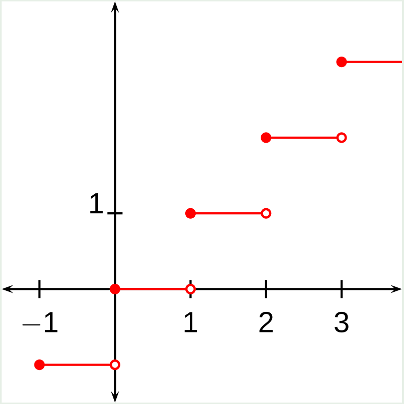

- $f(1)$ exists ($f(1) = 1$).
- $\lim_{x\to 1^+} f(x)$
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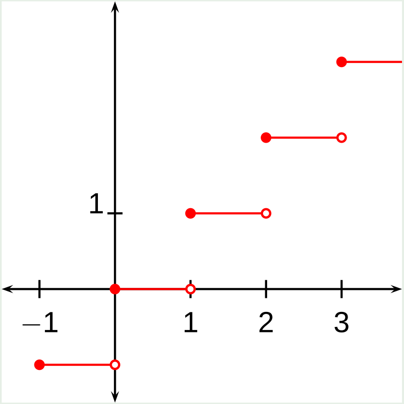

- $f(1)$ exists ($f(1) = 1$).
- $\lim_{x \to 1^+} f(x)$
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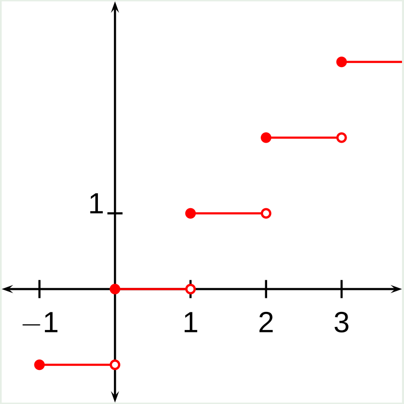

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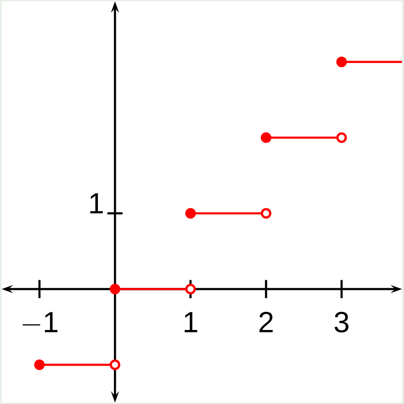

- $f(1)$ exists ($f(1) = 1$).
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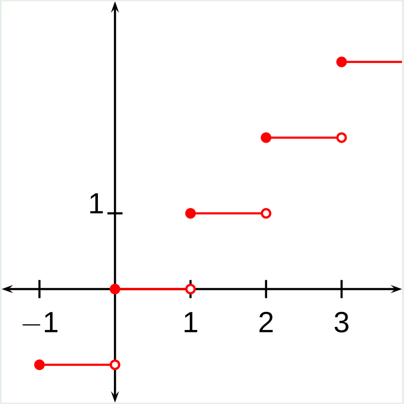

- $f(1)$ exists ($f(1) = 1$).
- $\lim_{x \to 1^+} f(x) = 1$.
- $\lim_{x \to 1^-} f(x) = 0$.
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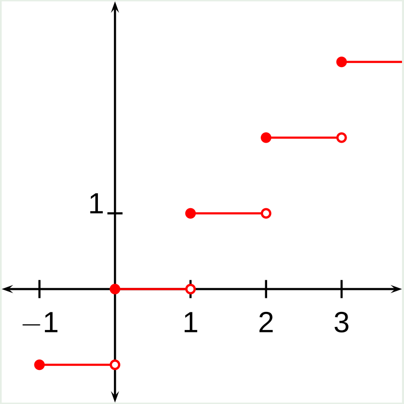

- $f(1)$ exists ($f(1) = 1$).
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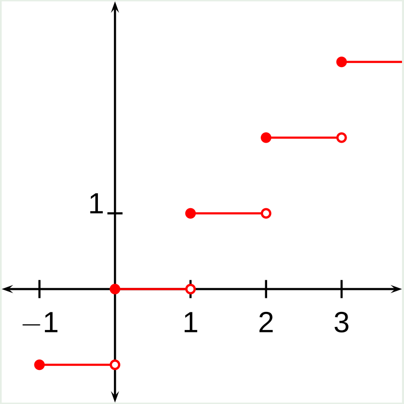

- $f(1)$ exists ($f(1) = 1$).
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- $\lim_{x \to 1} f(x)$ doesn't exist.

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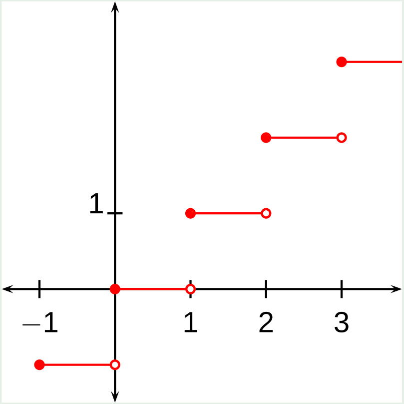

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- $\lim_{x \to 1^-} f(x) = 0$.
- $\lim_{x \to 1} f(x)$ doesn't exist.
- Discontinuous at 1.

Example

Where is this function discontinuous?

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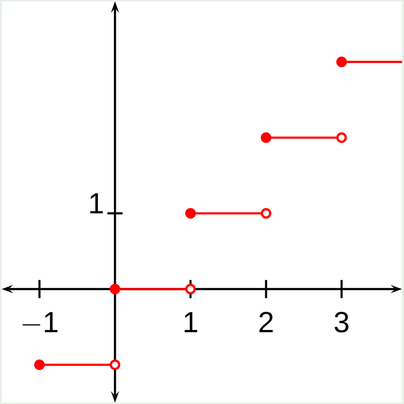

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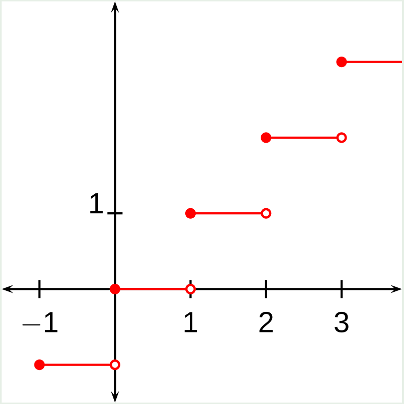
- $\lim_{x \to 1^-} f(x) = 0$.
- $\lim_{x \to 1} f(x)$ doesn't exist.
- Discontinuous at 1.
- Discontinuous at every integer n.

Example

Where is this function discontinuous?

$$f(x) = \lfloor x \rfloor$$



- $f(1)$ exists ($f(1) = 1$).
- $\lim_{x \to 1^+} f(x) = 1$.
- $\lim_{x \to 1^-} f(x) = 0$.
- $\lim_{x \to 1} f(x)$ doesn't exist.
- Discontinuous at 1.
- Discontinuous at every integer n.
- These are called jump discontinuities because the function "jumps" at these numbers (i.e., the left limit doesn't equal the right limit).