Math 140

Lecture 6

Greg Maloney

with modifications by T. Milev

University of Massachusetts Boston

February 14, 2013

Outline

Definition (Continuous from the Right or Left)

A function f is continuous from the right at a number a if

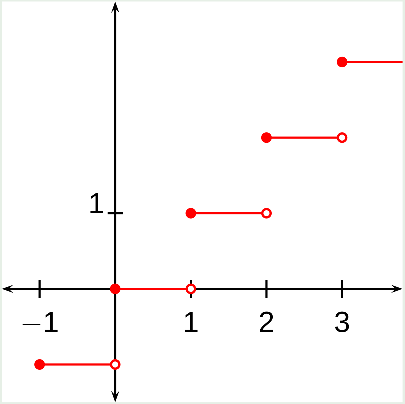
$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

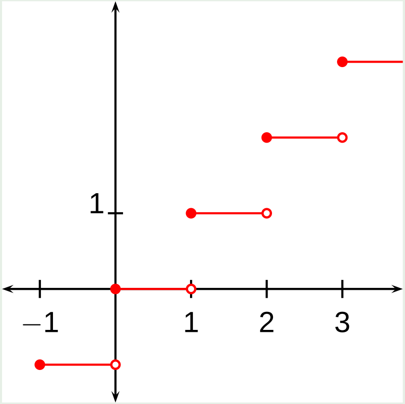
Example

Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



Example

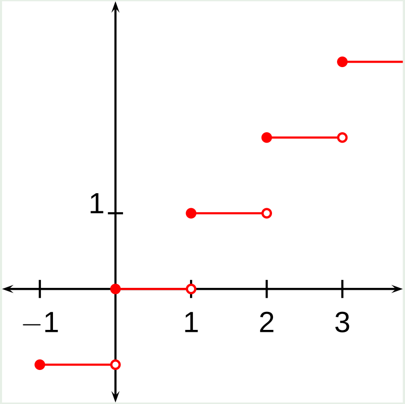
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) =$
- $\displaystyle\lim_{x\to n^+} f(x) =$
- $\displaystyle\lim_{x\to n^-} f(x) =$

Example

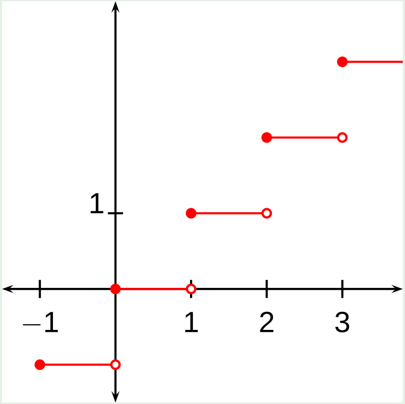
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) =$
- $\lim_{x \to n^+} f(x) =$
- $\lim_{x \to n^-} f(x) =$

Example

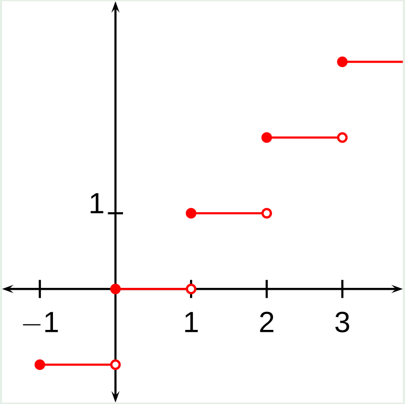
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim_{x \to n^+} f(x) =$
- $\lim_{x \to n^-} f(x) =$

Example

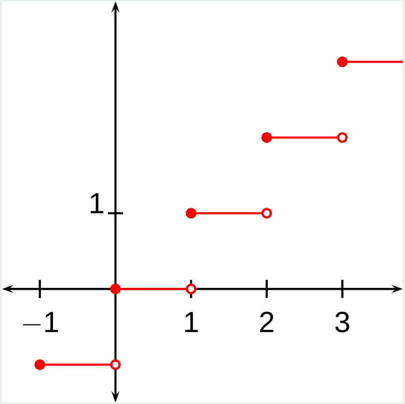
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim\limits_{x \to n^+} f(x) =$
- $\lim\limits_{x \to n^-} f(x) =$

Example

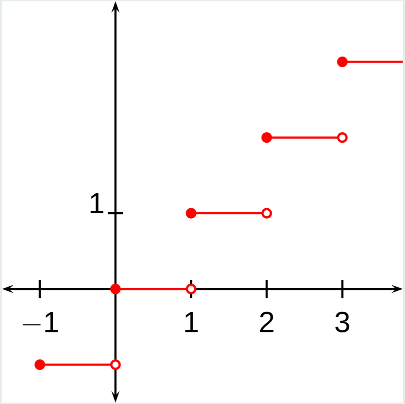
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim\limits_{x \to n^+} f(x) = n$.
- $\lim\limits_{x \to n^-} f(x) =$

Example

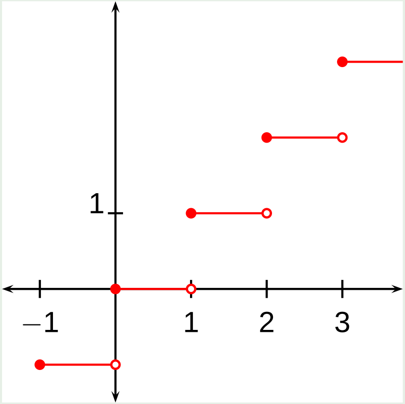
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim_{x \to n^+} f(x) = n$.
- Continuous from the right at n.
- $\lim_{x \to n^-} f(x) =$

Example

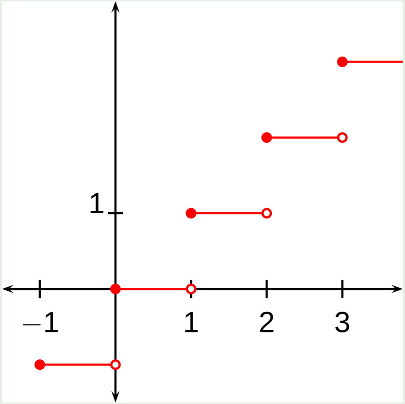
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim_{x \to n^+} f(x) = n$.
- Continuous from the right at n.
- $\lim_{x \to n^-} f(x) =$

Example

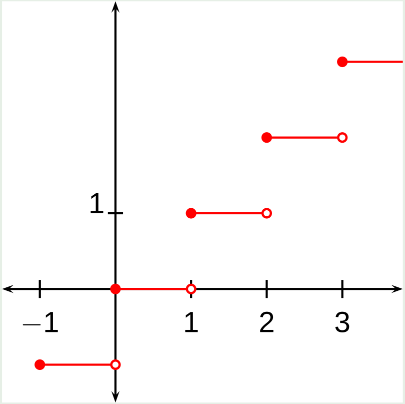
Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim_{x \to n^+} f(x) = n$.
- Continuous from the right at n.
- $\lim_{x \to n^-} f(x) = n - 1$.

Example

Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim_{x \to n^+} f(x) = n$.
- Continuous from the right at n.
- $\lim_{x \to n^-} f(x) = n - 1$.
- Discontinuous from the left at n.

Definition (Continuous on an Interval)

A function f is continuous on an interval if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, continuous at the endpoint means continuous from the right or continuous from the left.

Definition (Continuous on an Interval)

A function f is continuous on an interval if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, continuous at the endpoint means continuous from the right or continuous from the left. This is a consequence from the definition of continuity. As an exercise, you may try to reason why.

Definition (Continuous on an Interval)

A function f is continuous on an interval if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, continuous at the endpoint means continuous from the right or continuous from the left. This is a consequence from the definition of continuity. As an exercise, you may try to reason why.

Think of a function that is continuous on an interval as a function that has no breaks in its graph, and so can be drawn "without lifting your pen".

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a)\neq 0$.

Proof.

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a) \neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

$$\lim_{x\to a}(f+g)(x)$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a) \neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a}[f(x)+g(x)]$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$.

Proof.

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

$$\begin{aligned} \lim_{x \to a} (f+g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \end{aligned}$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a) \neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

$$\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{(by Law 1)} \\
&= f(a) + g(a)
\end{aligned}$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$.

Proof.

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

$$\begin{aligned}
\lim_{x \to a} (f+g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\
&= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\
&= f(a) + g(a)
\end{aligned}$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a)\neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

$$\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{(by Law 1)} \\
&= f(a)+g(a) = (f+g)(a)
\end{aligned}$$

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

$$\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{(by Law 1)} \\
&= f(a)+g(a) = (f+g)(a)
\end{aligned}$$

This shows $f+g$ is continuous at a.

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a) \neq 0$.

Proof.

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

$$\begin{aligned} \lim_{x \to a} (f+g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a) + g(a) = (f+g)(a) \end{aligned}$$

This shows $f + g$ is continuous at a. The other parts are similar. □

Theorem (Classes of Continuous Functions)

The following types of functions are continuous at every number in their domains:

polynomials	*rational functions*
root functions	*trigonometric functions*

Theorem (Compositions of Continuous Functions)

If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example

Find $\lim\limits_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

Example

Find $\lim\limits_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain.

Example

Find $\lim\limits_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

Example

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Example

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

$$\begin{aligned} & \lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \\ = & \lim_{x \to -2} f(x) \end{aligned}$$

Example

Find $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

$$
\begin{aligned}
& \lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} \\
= & \lim_{x\to -2} f(x) \\
= & f(-2)
\end{aligned}
$$

Example

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

$$
\begin{aligned}
& \lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \\
= & \lim_{x \to -2} f(x) \\
= & f(-2) \\
= & \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}
\end{aligned}
$$

Example

Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is everything but $5/3$.

$$\begin{aligned}
&\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \\
= &\lim_{x \to -2} f(x) \\
= &f(-2) \\
= &\frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \\
= &\frac{-1}{11}
\end{aligned}$$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) =$
- $h(x) =$
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) =$
- $h(x) =$
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) =$
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) =$
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) =$
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) =$

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) = \frac{1}{x}$.

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) = \frac{1}{x}$.
- These functions are continuous on their domains, so F is continuous on its domain.

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) = \frac{1}{x}$.
- These functions are continuous on their domains, so F is continuous on its domain.
- Its domain is

Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) = \frac{1}{x}$.
- These functions are continuous on their domains, so F is continuous on its domain.
- Its domain is everything but 3 and -3.

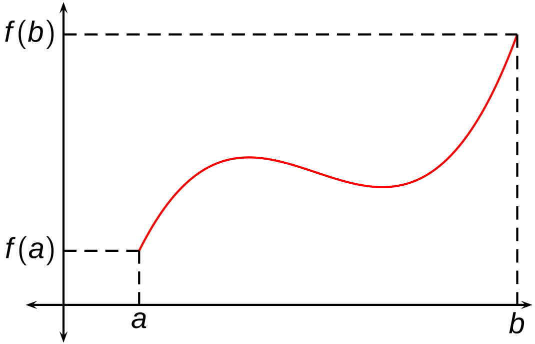
Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(x) = \sqrt{x}$.
- $g(x) = x - 4$.
- $f(x) = \frac{1}{x}$.
- These functions are continuous on their domains, so F is continuous on its domain.
- Its domain is everything but 3 and -3.
- Therefore F is continuous on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

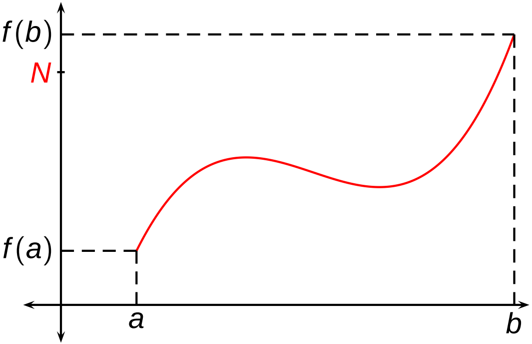
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



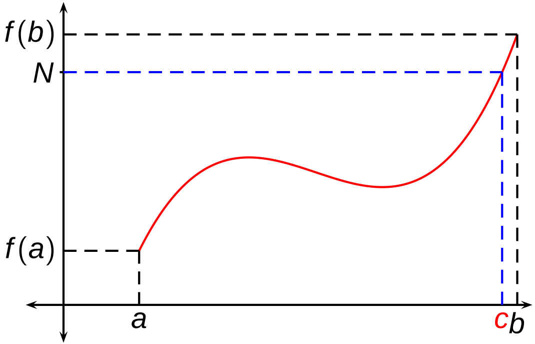
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



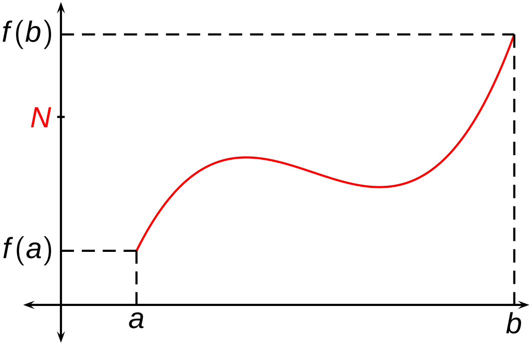
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



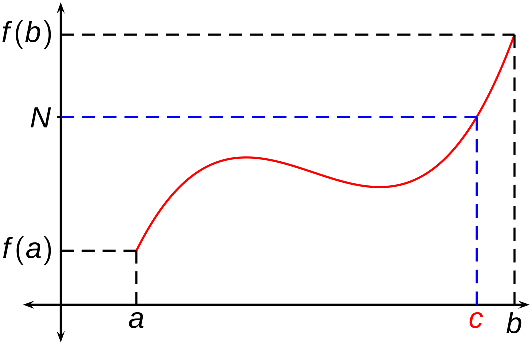
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



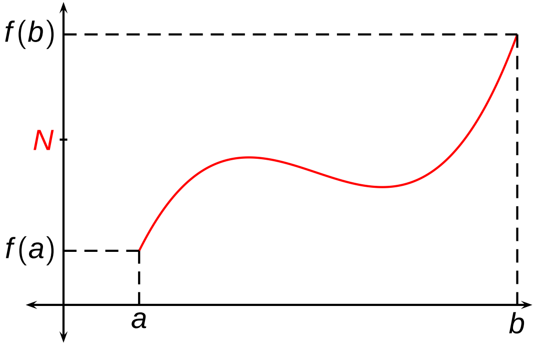
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



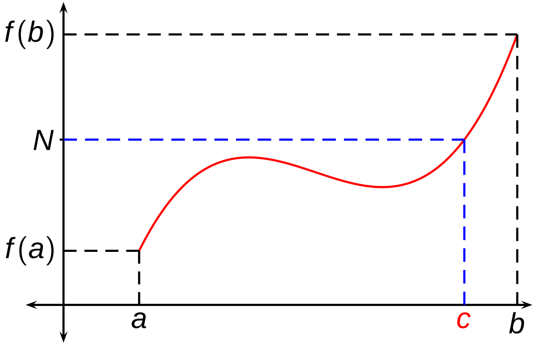
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



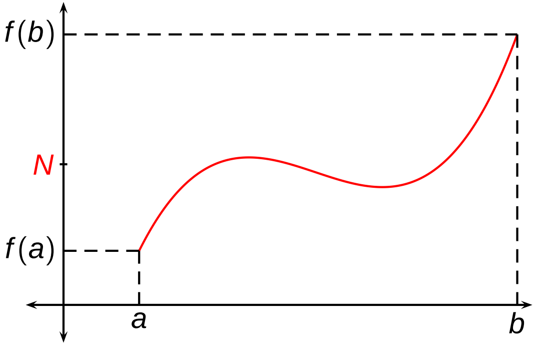
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



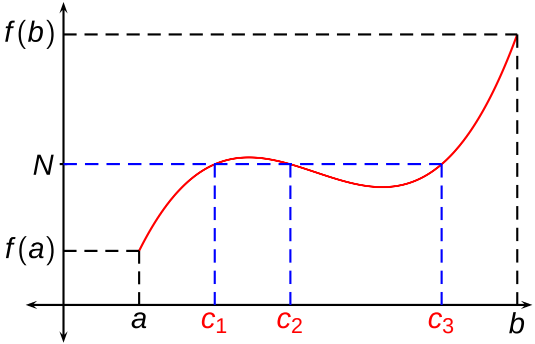
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- Use the intermediate value theorem with $a = 1$, $b = 2$, and $N = 0$.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- Use the intermediate value theorem with $a = 1$, $b = 2$, and $N = 0$.
- $f(1) = 4 \cdot 1^3 - 6 \cdot 1^2 + 3 \cdot 1 - 2 = -1 < 0$.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- Use the intermediate value theorem with $a = 1$, $b = 2$, and $N = 0$.
- $f(1) = 4 \cdot 1^3 - 6 \cdot 1^2 + 3 \cdot 1 - 2 = -1 < 0$.
- $f(2) = 4 \cdot 2^3 - 6 \cdot 2^2 + 3 \cdot 2 - 2 = 12 > 0$.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- Use the intermediate value theorem with $a = 1$, $b = 2$, and $N = 0$.
- $f(1) = 4 \cdot 1^3 - 6 \cdot 1^2 + 3 \cdot 1 - 2 = -1 < 0$.
- $f(2) = 4 \cdot 2^3 - 6 \cdot 2^2 + 3 \cdot 2 - 2 = 12 > 0$.
- $N = 0$ is a number between $f(1) = -1$ and $f(2) = 12$.

Example

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- Use the intermediate value theorem with $a = 1$, $b = 2$, and $N = 0$.
- $f(1) = 4 \cdot 1^3 - 6 \cdot 1^2 + 3 \cdot 1 - 2 = -1 < 0$.
- $f(2) = 4 \cdot 2^3 - 6 \cdot 2^2 + 3 \cdot 2 - 2 = 12 > 0$.
- $N = 0$ is a number between $f(1) = -1$ and $f(2) = 12$.
- Therefore there is a c between 1 and 2 such that $f(c) = 0$.