Math 140
Lecture 7

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with modifications by T. Milev

University of Massachusetts Boston

February 19, 201

Outline

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(1.2) Exponential Functions

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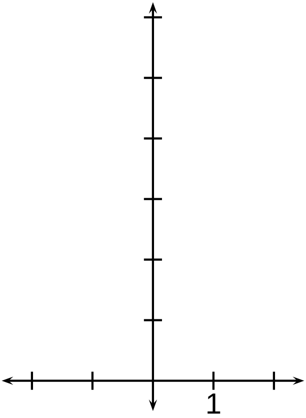

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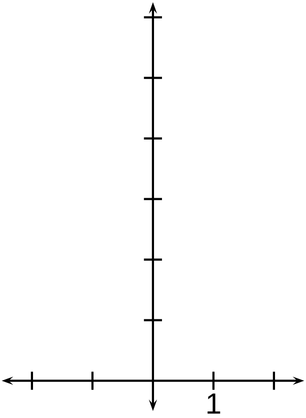

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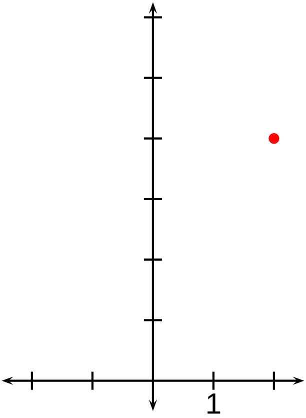

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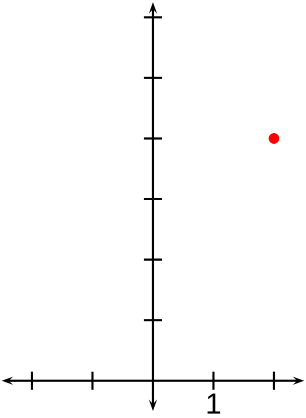

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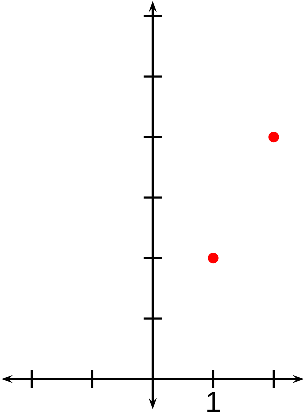

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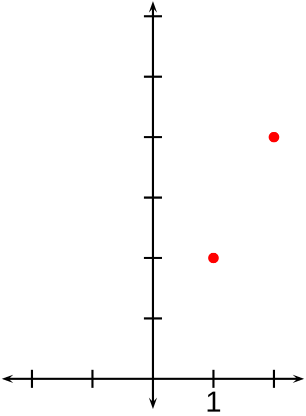

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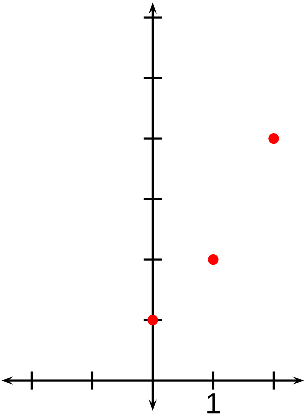

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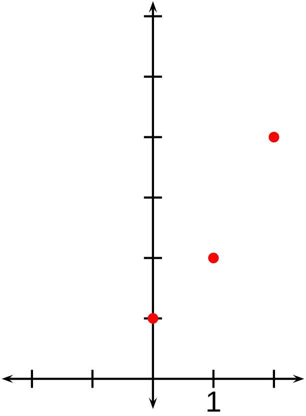

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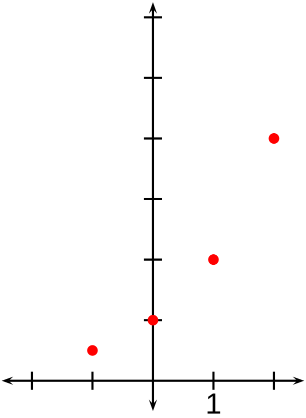

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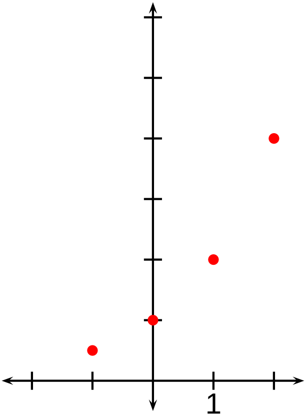

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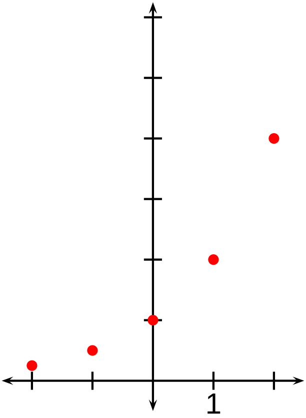

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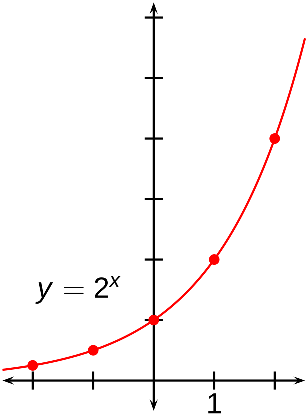

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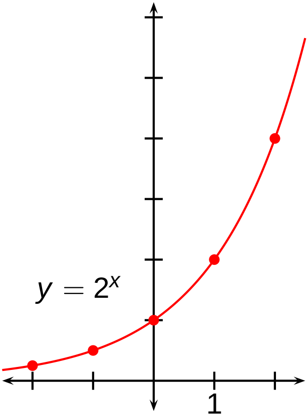

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Definition (Exponential Function)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

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- We discuss pros and cons.

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- This is the definition we assume in Calculus I.

Exponent definition approach II

- The following definition is equivalent, studied in Calculus II.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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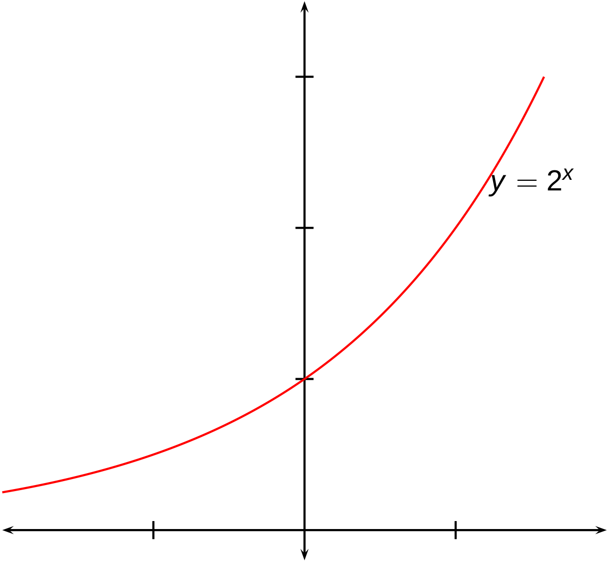
Graphs of various exponential functions.



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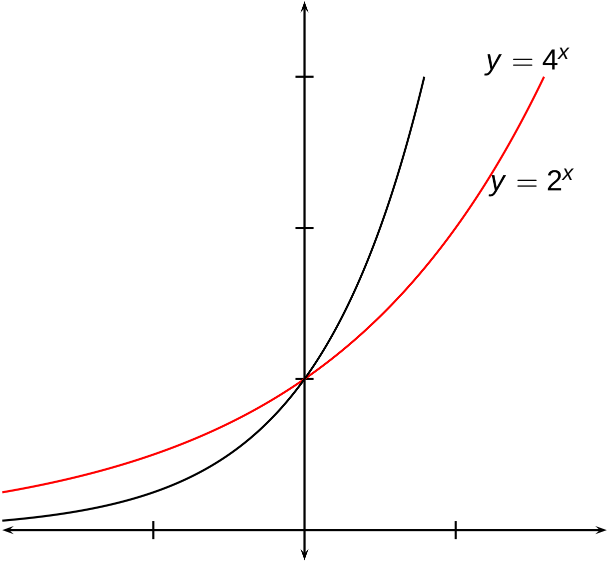

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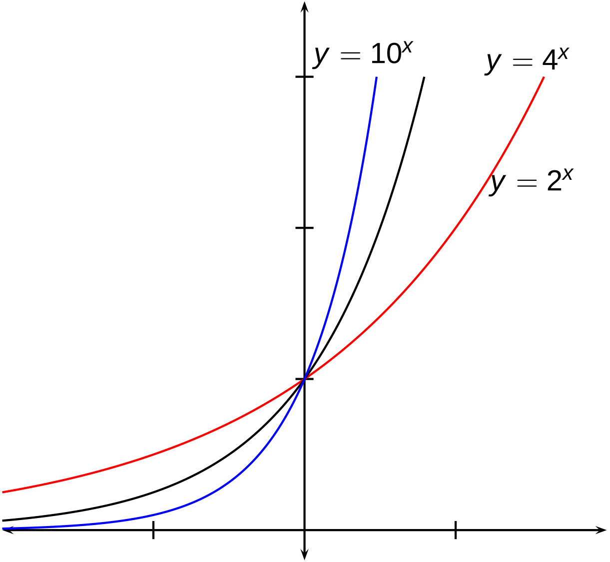

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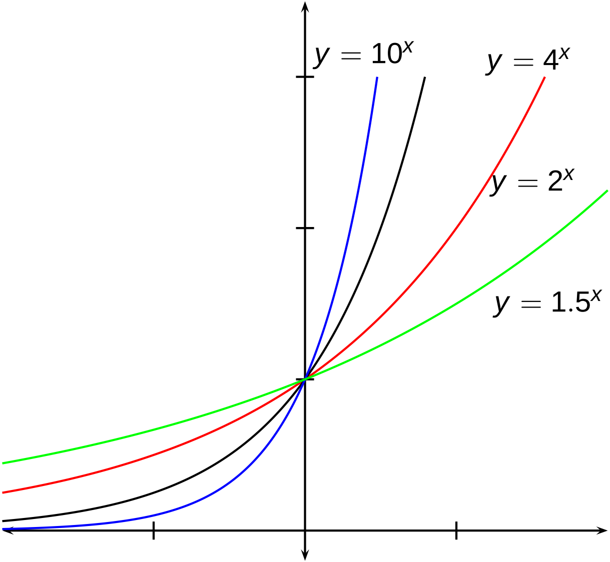

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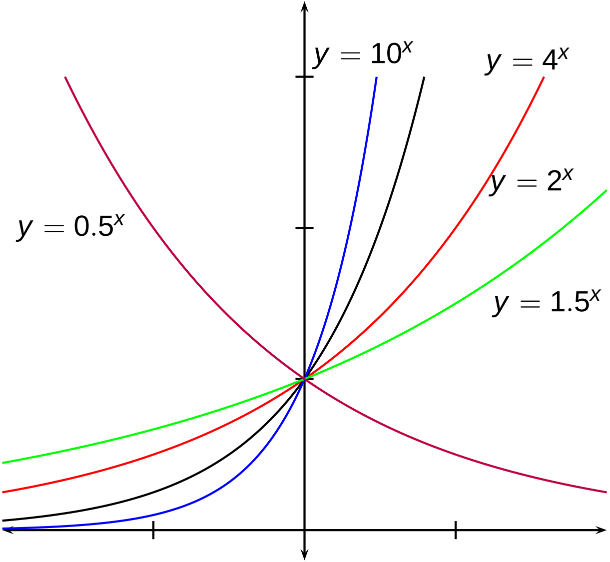

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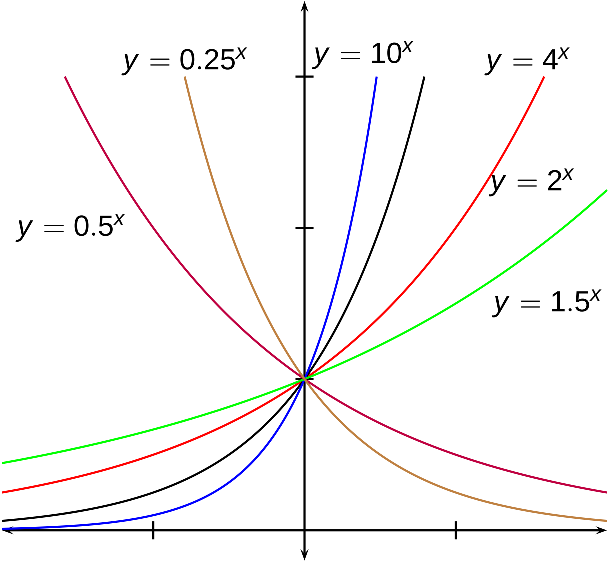
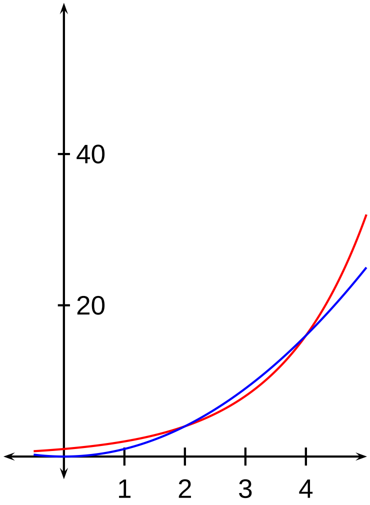


Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.



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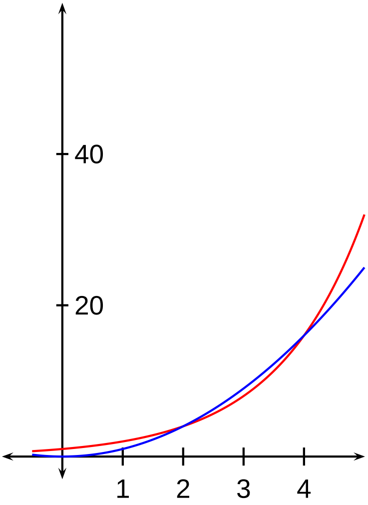

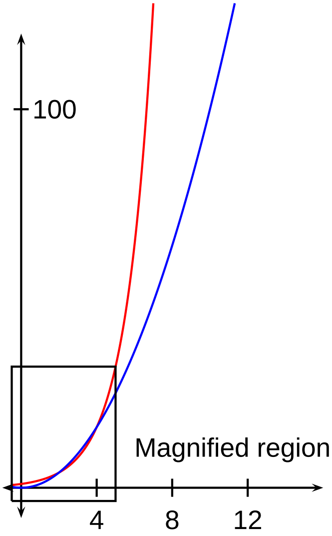

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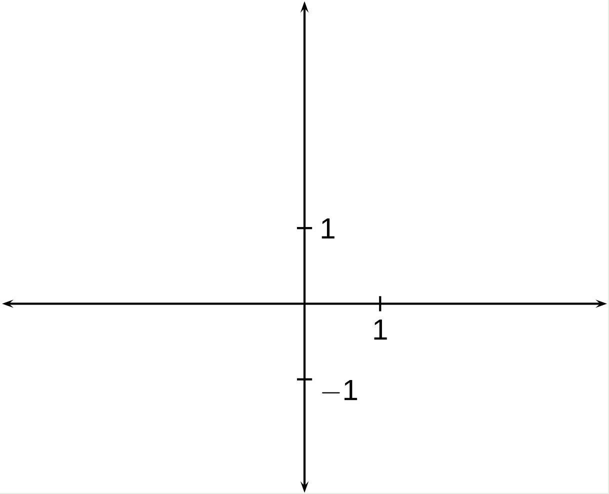

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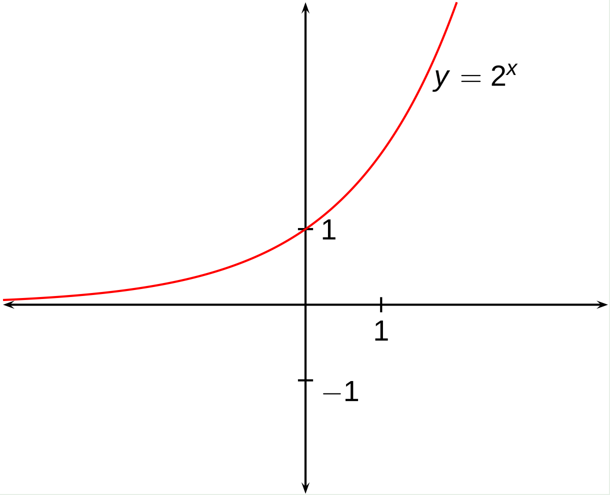

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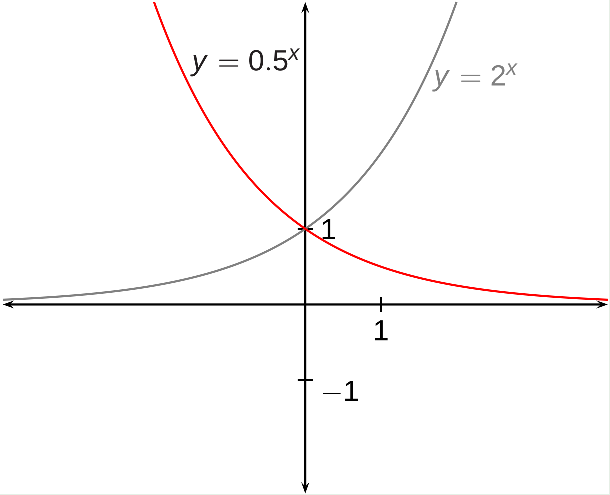

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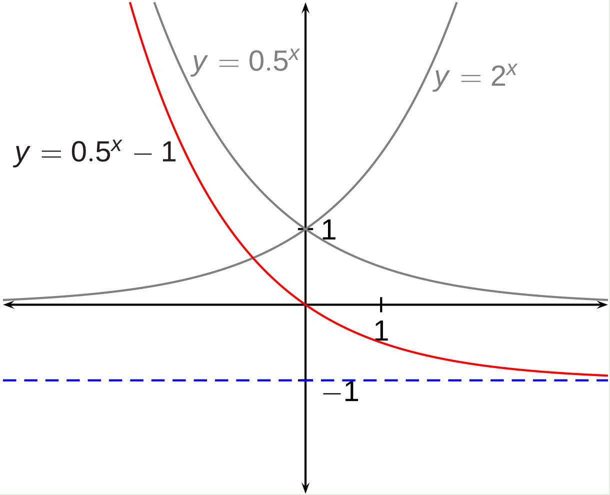
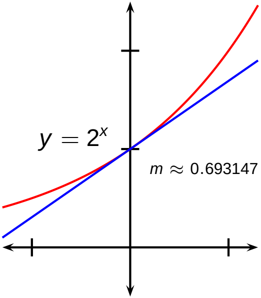


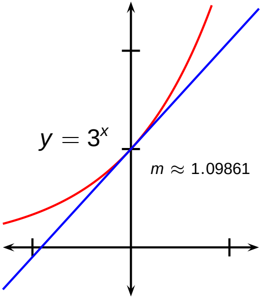
Recall from previous lectures.

- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

The Natural Exponential Function

- One base for an exponential function is especially useful.





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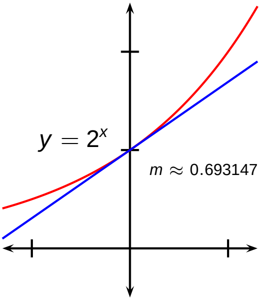

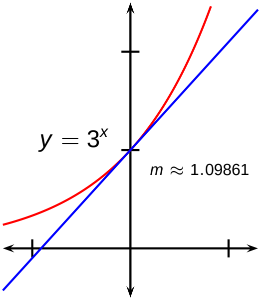

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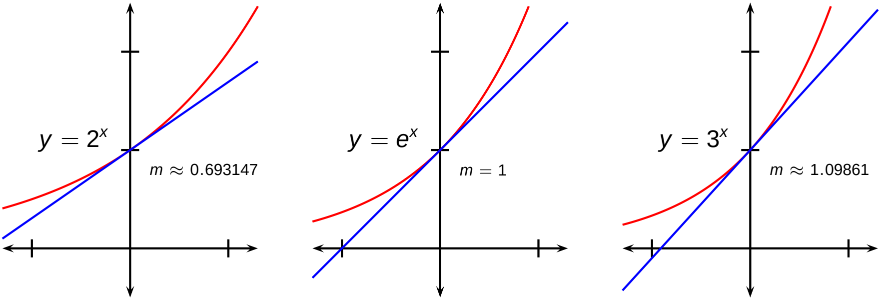
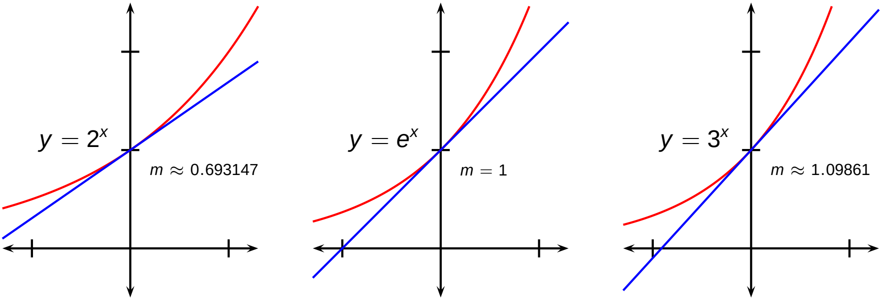


The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e.
- e is a number between 2 and 3. In fact, $e \approx 2.71828$.

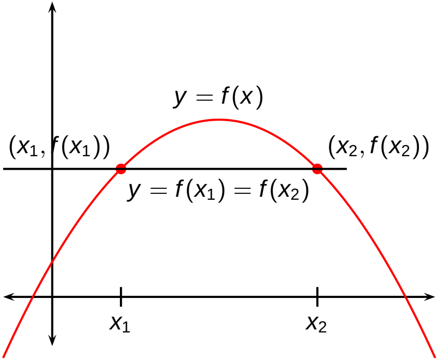


One-to-one Functions

Definition (One-to-one Function)

A function f is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

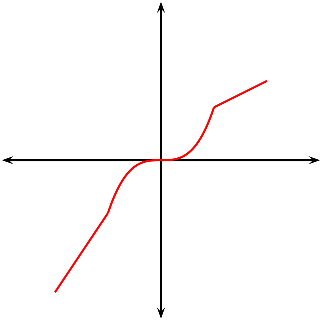


$\leftarrow$ This function is not one-to-one.

Question: How can we tell from the graph of a function whether it is one-to-one or not?
Answer: Use the horizontal line test.

The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once. □



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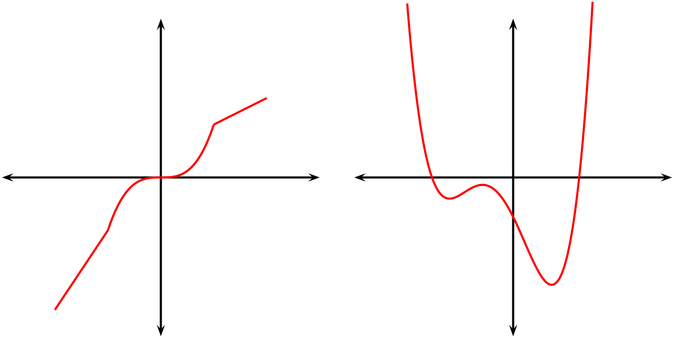

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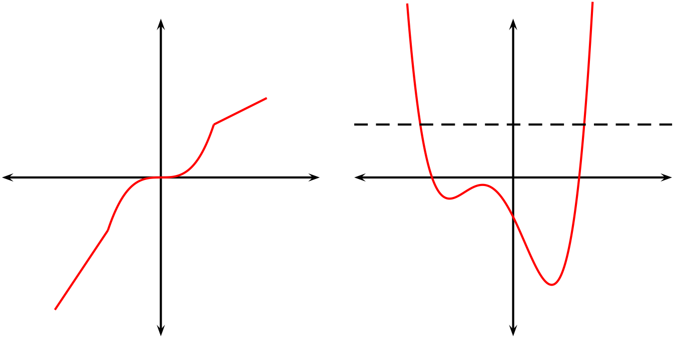

One-to-one

Not one-to-one

The Definition of the Inverse of f

Definition (f^{-1})

Let f be a one-to-one function with domain A and range B. Then the inverse of f is the function f^{-1} that has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y$$

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Example ($f(x) = x^3$)

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. This is because if $y = x^3$, then

$$f^{-1}(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x.$$

WARNING:
Do not mistake the -1 in $f^{-1}(x)$ for an exponent.

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}.$$

If you want to write $\frac{1}{f(x)}$ using exponents, you can write $(f(x))^{-1}$.

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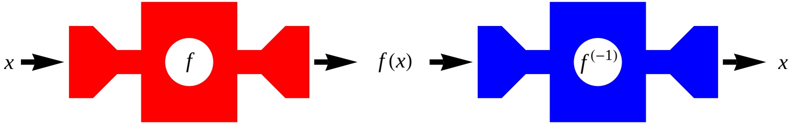

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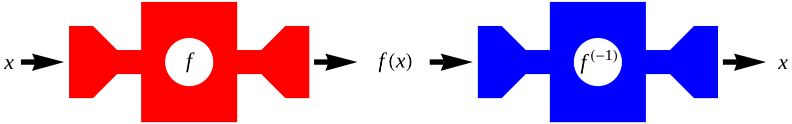

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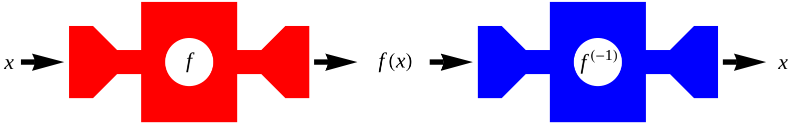

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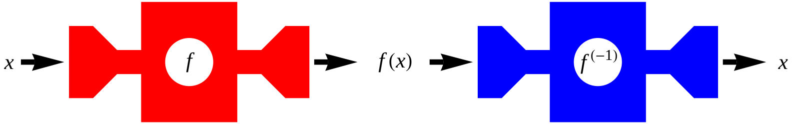

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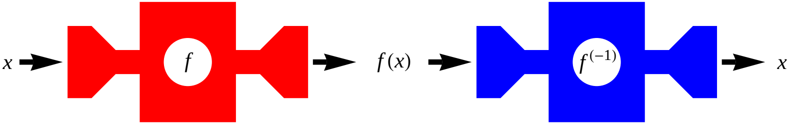

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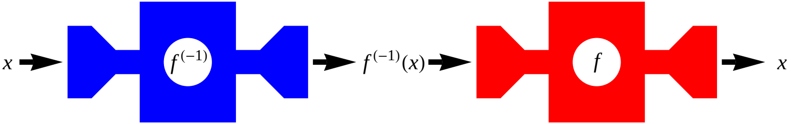

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Let $f(x) = 2x + \sin 2x + e^{x/2}$. Suppose I is a small interval around 0 such that f is one-to-one for $x \in I$. Find $f^{-1}(1)$.

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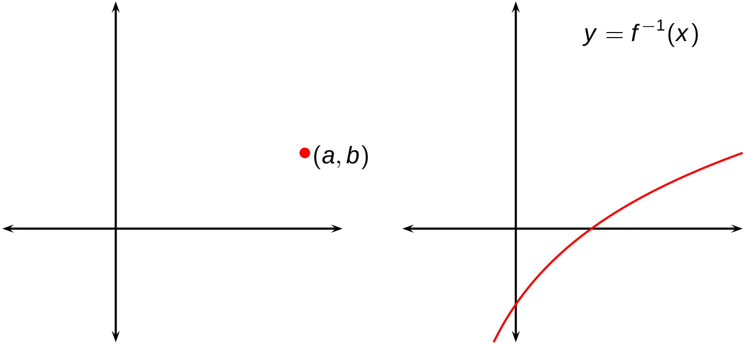
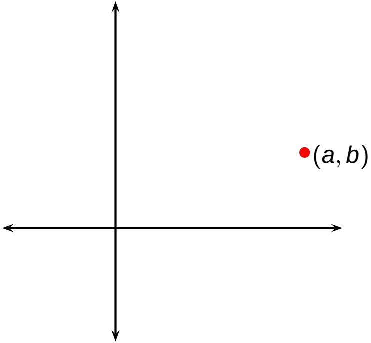
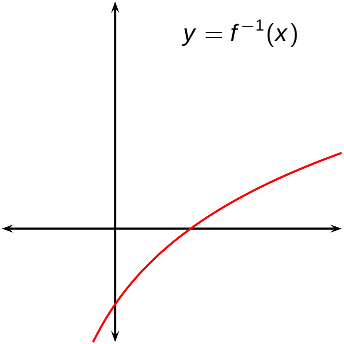


Interchanging x and y suggests a relation between the graphs of f^{-1} and f:





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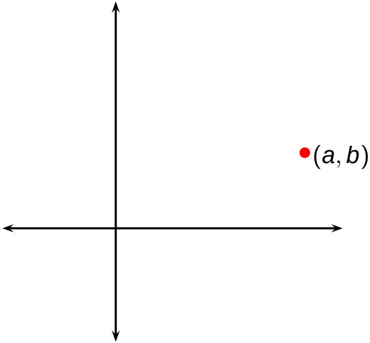

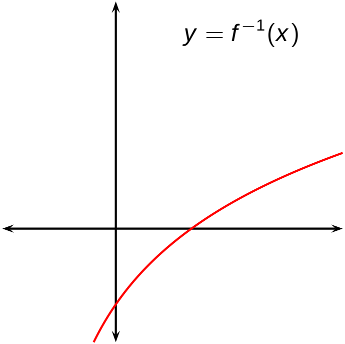

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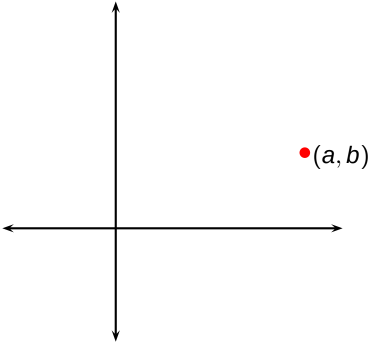

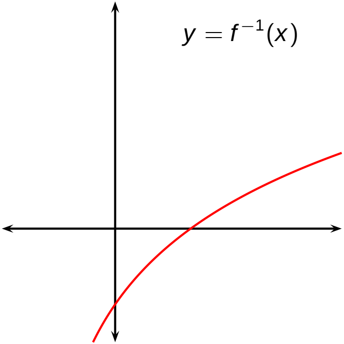

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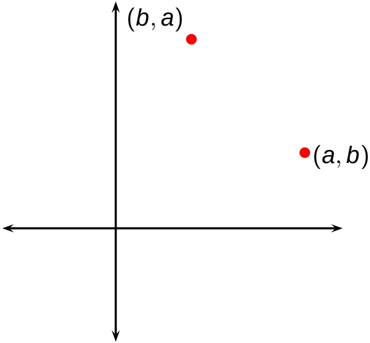

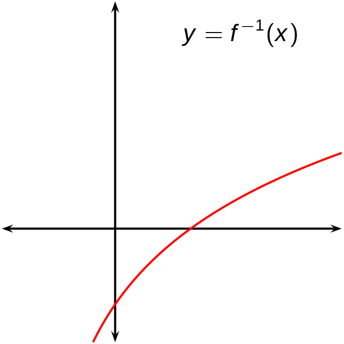

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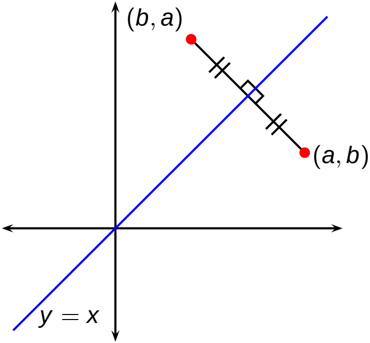

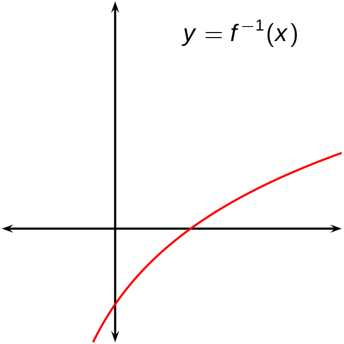

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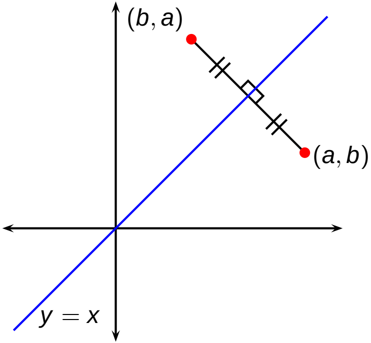

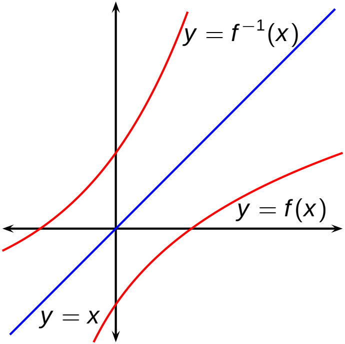

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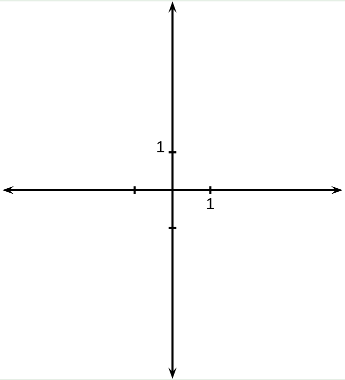
- (b, a) is the reflection of (a, b) in the line $y = x$.
- Therefore the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.

Example



Sketch the graph of $f(x) = \sqrt{-x-1}$ and its inverse function.

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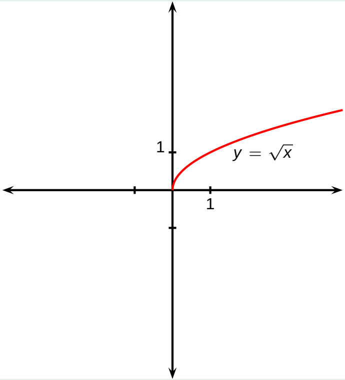

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- First draw the graph of $y = \sqrt{x}$.

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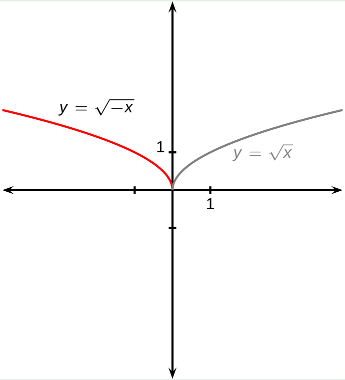

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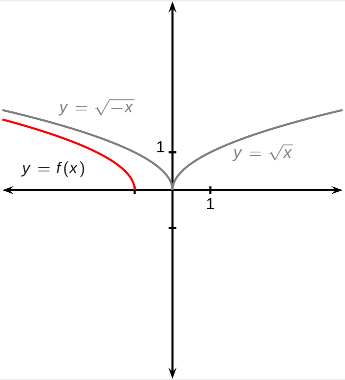

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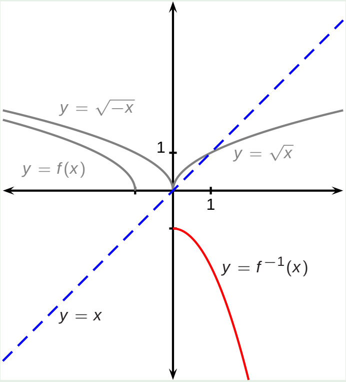

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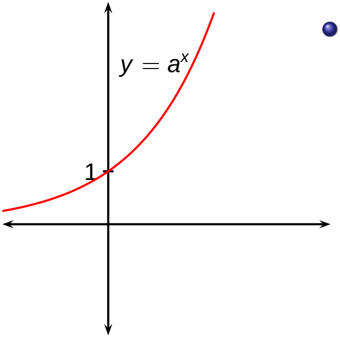
Logarithmic Functions



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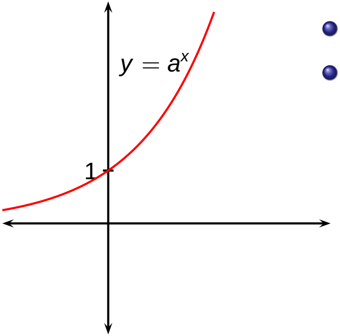

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- Let $f(x) = a^x$.

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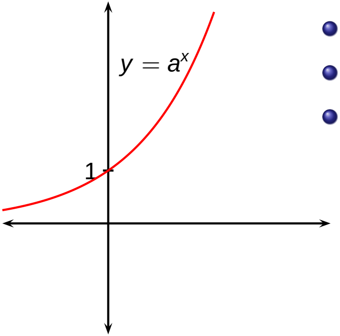

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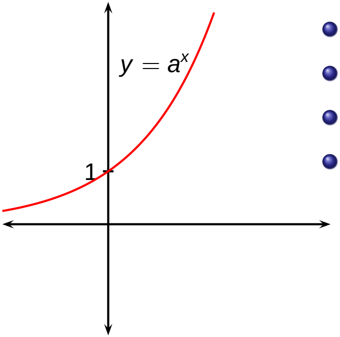

- Suppose $a > 0$, $a \neq 1$.
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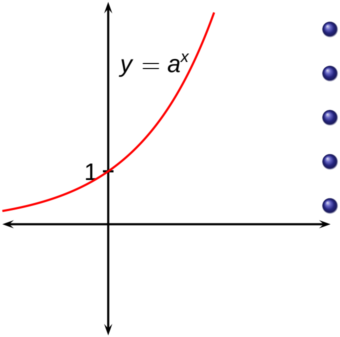
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.

Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
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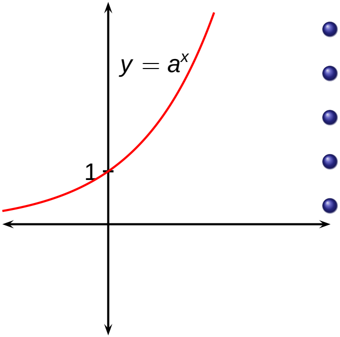

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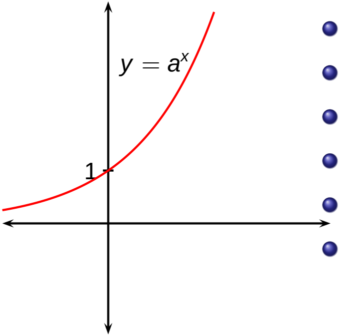

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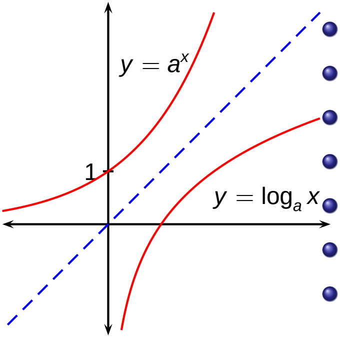

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If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

Evaluate:

1. $\log_3 81 =$
2. $\log_{25} 5 =$
3. $\log_{10} 0.001 =$

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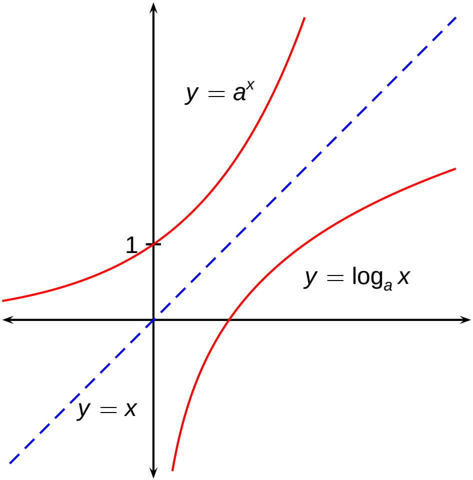

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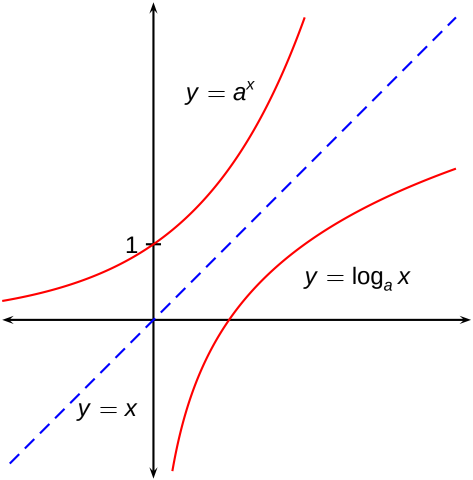

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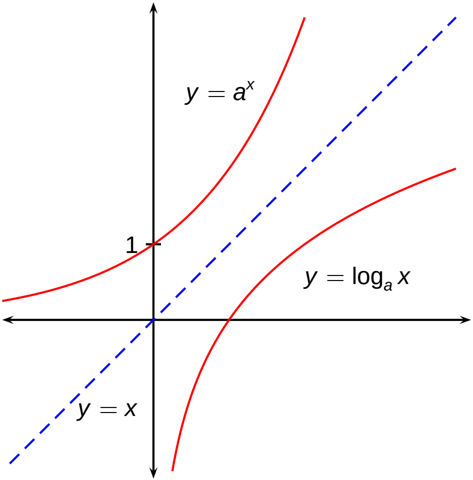

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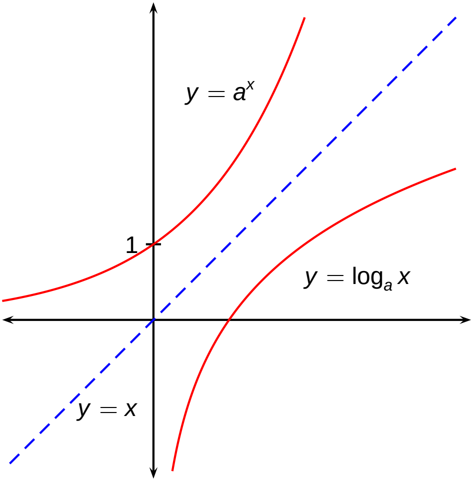

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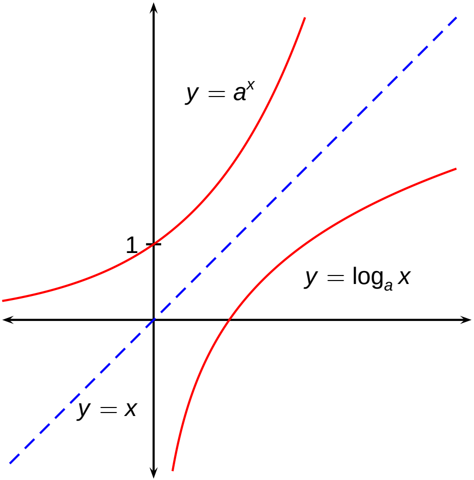

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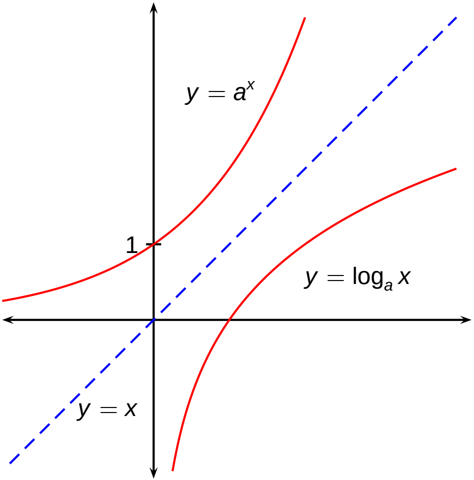

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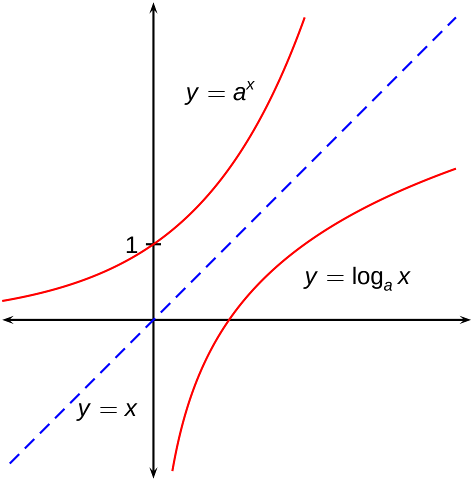

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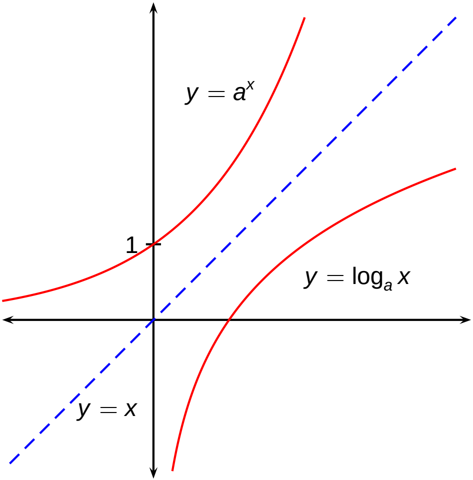

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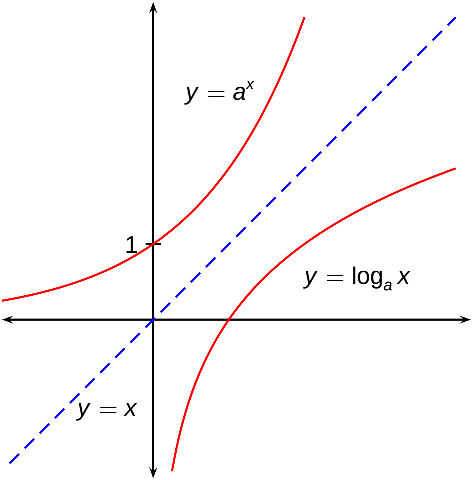

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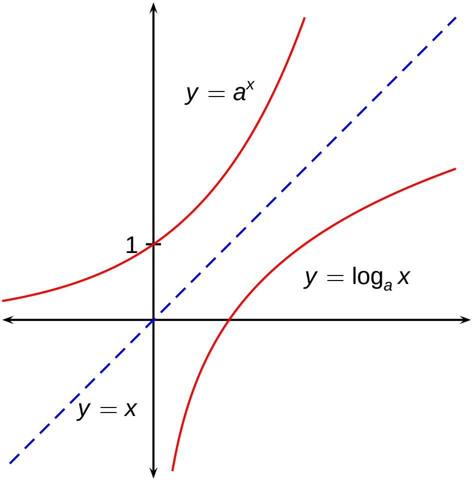

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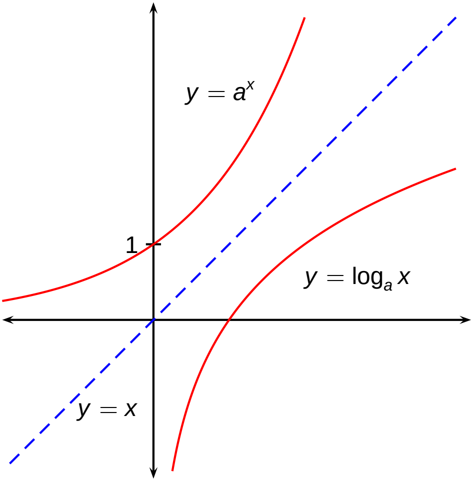

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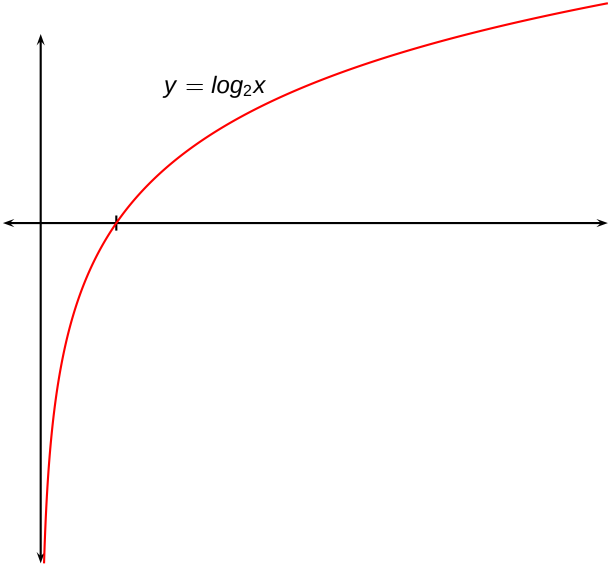
- Range of $\log_a x$: $\mathbb{R}$.
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$



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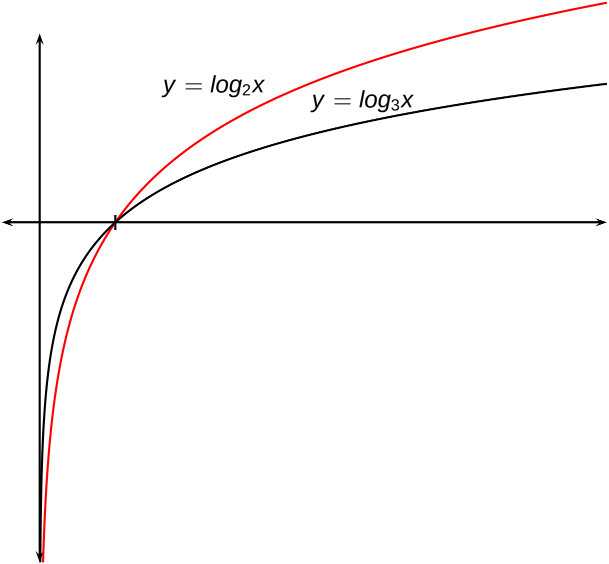

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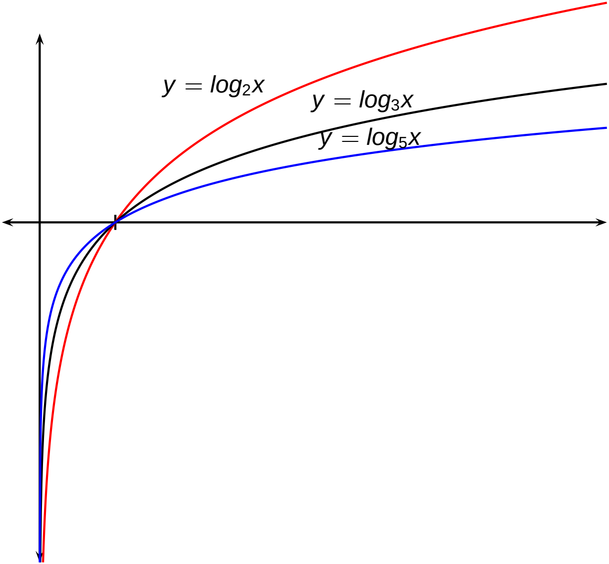

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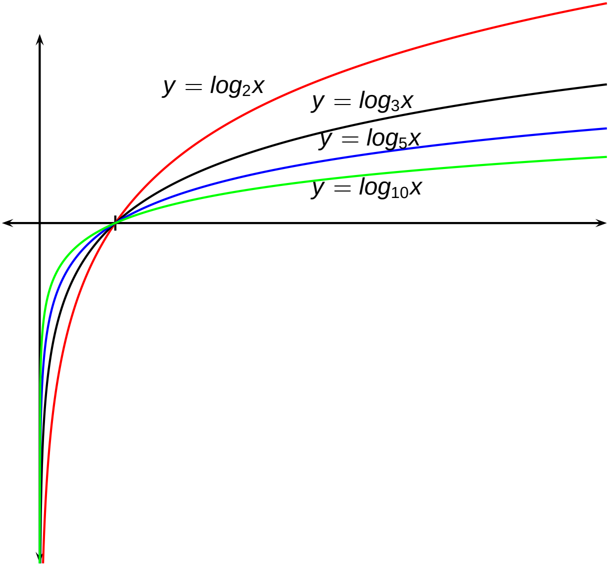

Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_{\frac{1}{a}} x = -\log_a x$
5. $\log_a b = \frac{1}{\log_b a}$
6. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$

Example

Use the properties of logarithms to evaluate the following:

$$\log_4 2 + \log_4 32 \qquad\qquad \log_2 80 - \log_2 5$$

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(because $4^3 = 64$.)

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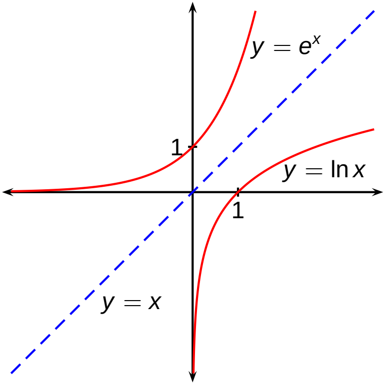
Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



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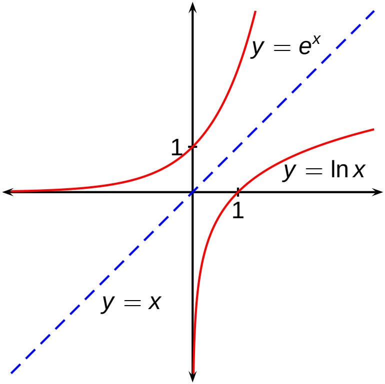

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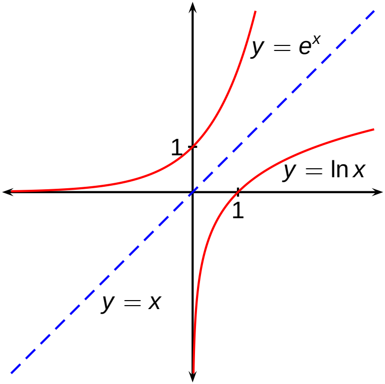

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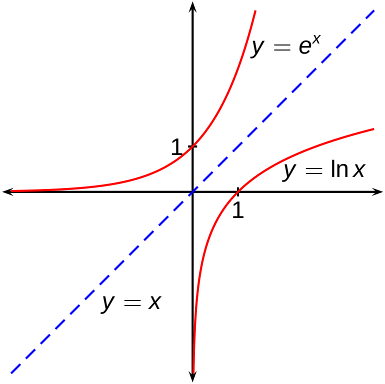

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Example

Solve the equation $e^{5-3x} = 10$

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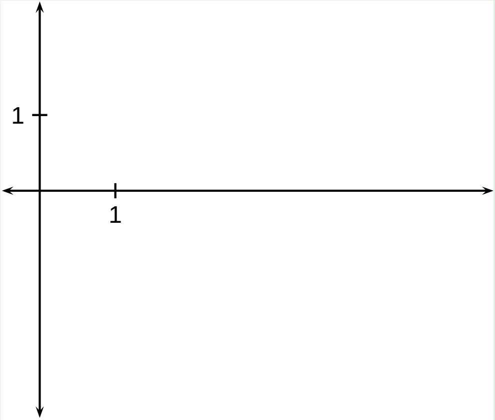
\text{Calculator:}\quad x &\approx 0.8991.
\end{aligned}$$

Example

Draw the graph of $y = \ln(x - 2) - 1$.



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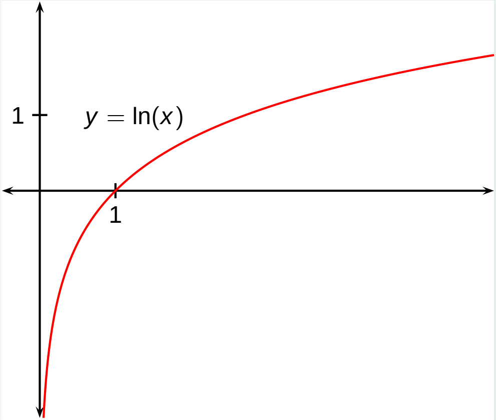

- Graph $y = \ln(x)$ assumed given.

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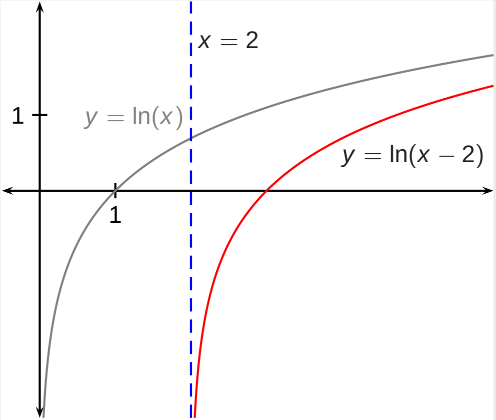

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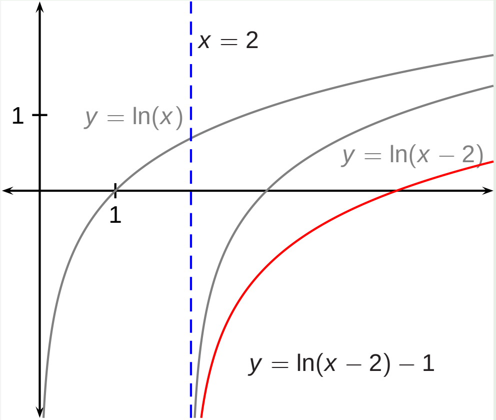

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- $g(x) - 1$ shifts graph 1 unit down.