Math 140

Lecture 8

Greg Maloney

with modifications by T. Milev

University of Massachusetts Boston

February 26, 2013

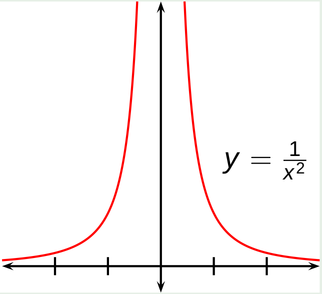
Outline

1. (3.4)Limits Involving Infinity
   - Infinite Limits
   - Limits at Infinity; Horizontal Asymptotes
   - Infinite Limits at Infinity

Infinite Limits

Example

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.



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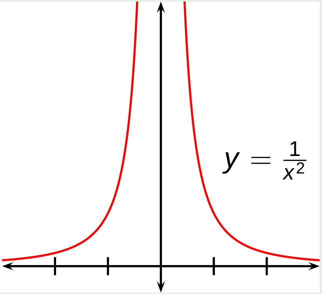

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
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± 0.01	10,000
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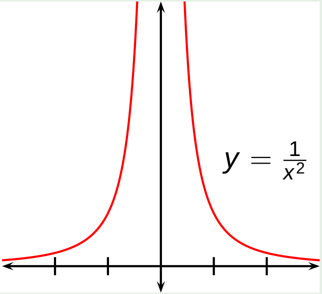

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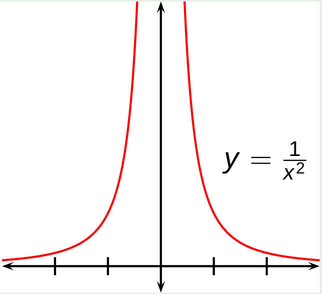

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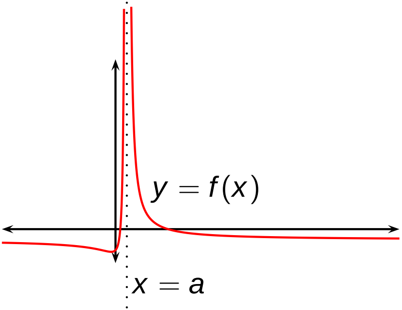
- As x gets close to 0, so does x^2, so $1/x^2$ gets large.
- $1/x^2$ can be made arbitrarily large by taking x close enough to 0.
- $f(x)$ doesn't approach a number, so $\lim_{x \to 0} 1/x^2$ doesn't exist.

Definition (Infinite Limit)

Let f be a function defined on both sides of a, except perhaps at a. Then

$$\lim_{x \to a} f(x) = \infty$$

means the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.



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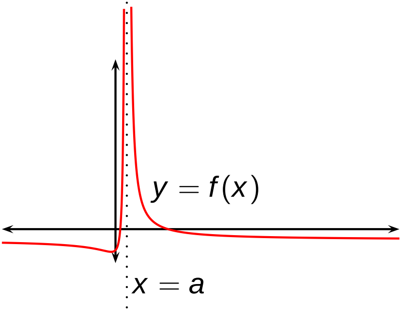

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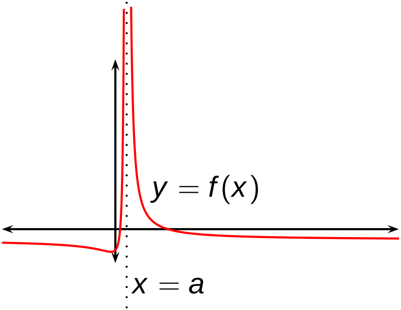

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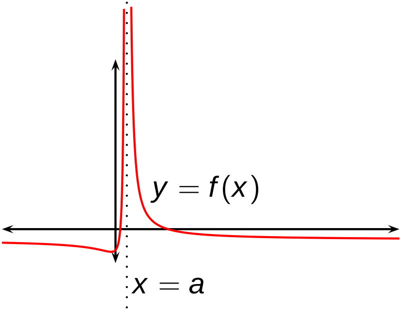

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- In such cases, the limit does not exist.
- ∞ is not a number. The notation $\lim_{x \to a} f(x) = \infty$ expresses the particular way in which the limit doesn't exist.

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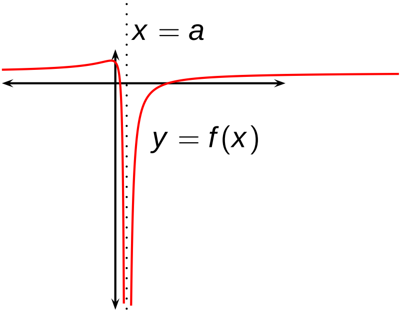

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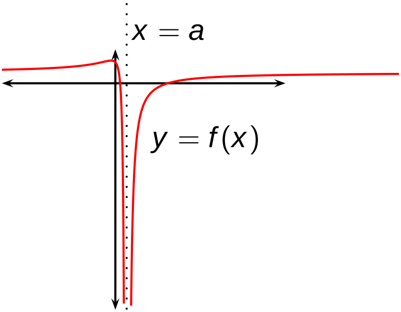

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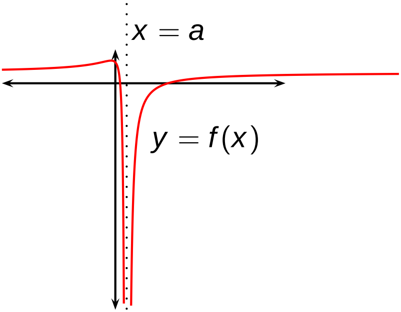

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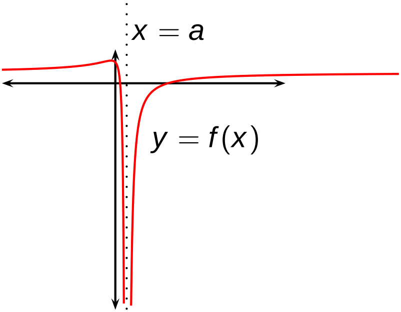

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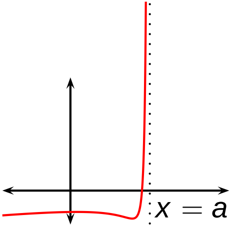
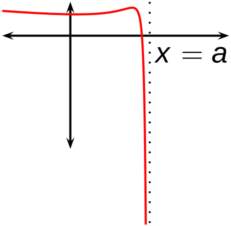
There are similar definitions for one-sided limits:

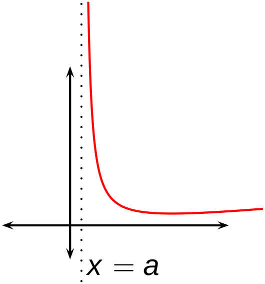


$\lim_{x \to a^-} f(x) = \infty$

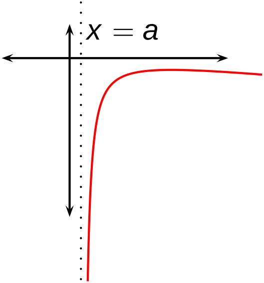


$\lim_{x \to a^-} f(x) = -\infty$

$x \to a^-$ means we only consider $x < a$.



$\lim_{x \to a^+} f(x) = \infty$



$\lim_{x \to a^+} f(x) = -\infty$

$x \to a^+$ means we only consider $x > a$.

Definition (Vertical Asymptote)

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
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For instance, the y-axis is a vertical asymptote for $f(x) = 1/x^2$ because $\lim_{x\to 0} f(x) = \infty$.

Example

Find $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$.

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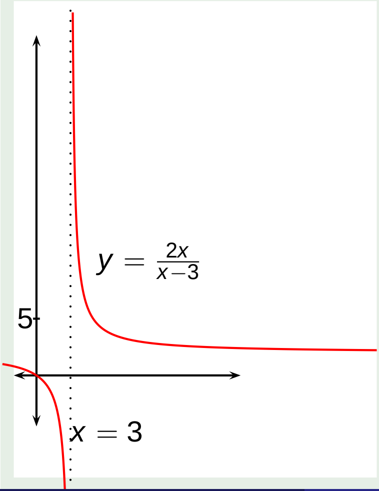

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- So $2x/(x-3)$ is a negative number with large absolute value.
- $x = 3$ is a vertical asymptote for $f(x) = 2x/(x-3)$.

$$\lim_{x \to a} f(x)$$

If we plug in a and get

$$f(a) = \frac{\text{something different from 0}}{0},$$

then the limit will be DNE, ∞, or $-\infty$.
To determine what the answer is, this is what we do:

1. Factor.
2. Determine if each factor is positive or negative.
3. An odd number of negative factors means the limit is $-\infty$.
4. An even number of negative factors means the limit is ∞.
5. For a two-sided limit, the answer is DNE unless the left limit and the right limit are either both ∞ or both $-\infty$.

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$$\text{Plug in 1:} \quad \frac{(1)^2-3(1)}{(1)^2-3(1)+2} = \frac{-2}{0}$$

The numerator is non-zero and the denominator is zero. Therefore the answer is DNE, ∞, or $-\infty$.

$$\text{Factor:} \quad \lim_{x\to 1^+} \frac{x^2-3x}{x^2-3x+2} = \lim_{x\to 1^+} \frac{x(x-3)}{(x-2)(x-1)}$$

$$\to \frac{(+)(-)}{(-)(+)}$$

$$= (+)$$

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x^2 - 3x + 2}$$

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$$\text{Therefore} \quad \lim_{x \to 1^+} \frac{x^2 - 3x}{x^2 - 3x + 2} = +\infty.$$

Example (Infinite Limit)

Find $$\lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x}$$

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x}$$

Plug in -1: $$\frac{(-1)^2+5(-1)+6}{(-1)^3+2(-1)^2+(-1)} = -$$

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x \to -1} \frac{x^2 + 5x + 6}{x^3 + 2x^2 + x}$$

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$$\text{Factor:} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} = \lim_{x\to -1} \text{\_\_\_\_\_\_\_\_\_\_}$$

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$$\text{Find} \quad \lim_{x \to -1} \frac{x^2 + 5x + 6}{x^3 + 2x^2 + x}$$

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The numerator is non-zero and the denominator is zero. Therefore the answer is DNE, ∞, or $-\infty$.

$$\text{Factor:} \quad \lim_{x \to -1} \frac{x^2 + 5x + 6}{x^3 + 2x^2 + x} = \lim_{x \to -1} \frac{(x+2)(x+3)}{}$$

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x \to -1} \frac{x^2 + 5x + 6}{x^3 + 2x^2 + x}$$

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The numerator is non-zero and the denominator is zero. Therefore the answer is DNE, ∞, or $-\infty$.

$$\text{Factor:} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} = \lim_{x\to -1} \frac{{\color{red}(x+2)}(x+3)}{x(x+1)^2}$$

$$\to \frac{{\color{red}(\quad)}(\quad)}{(\quad)(\quad)}$$

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x \to -1} \frac{x^2 + 5x + 6}{x^3 + 2x^2 + x}$$

$$\text{Plug in } -1: \quad \frac{(-1)^2 + 5(-1) + 6}{(-1)^3 + 2(-1)^2 + (-1)} = \frac{2}{0}$$

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$$=$$

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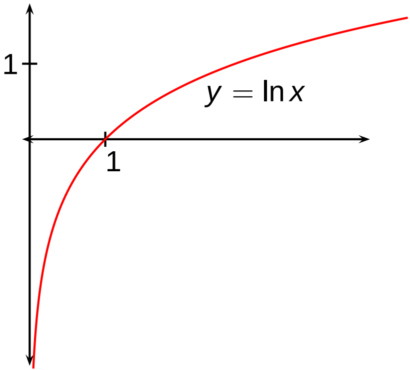
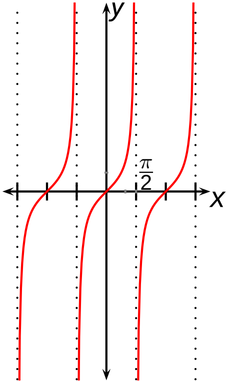
$$\begin{aligned} \text{Factor:} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} &= \lim_{x\to -1} \frac{(x+2)(x+3)}{x(x+1)^2} \\ &\to \frac{(+)(+)}{(-)(+)} \\ &= (-) \end{aligned}$$

$$\text{Therefore} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} = -\infty.$$





$\ln x$

$\lim_{x \to 0^+} \ln x =$

$\lim_{x \to 0^-} \ln x =$

$\lim_{x \to 0} \ln x =$

$\tan x$

$\lim_{x \to \pi/2^+} \tan x =$

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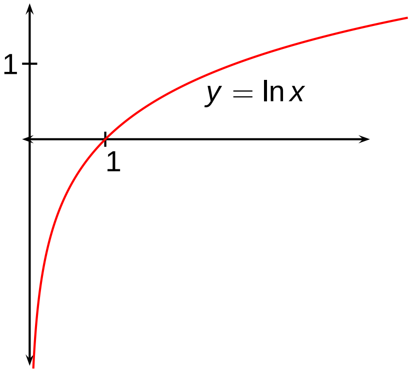

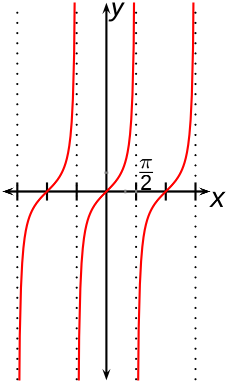

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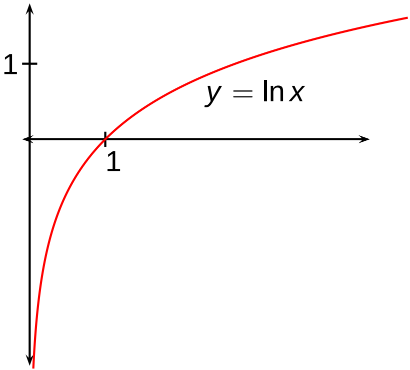

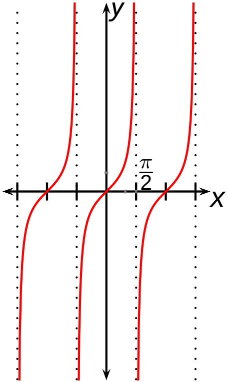

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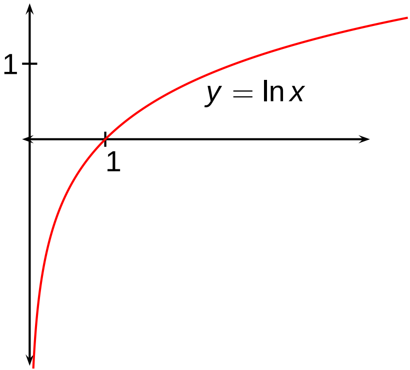

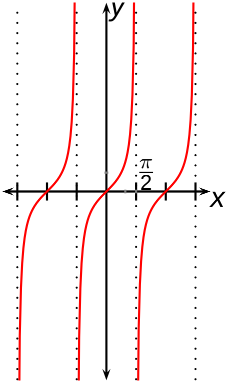

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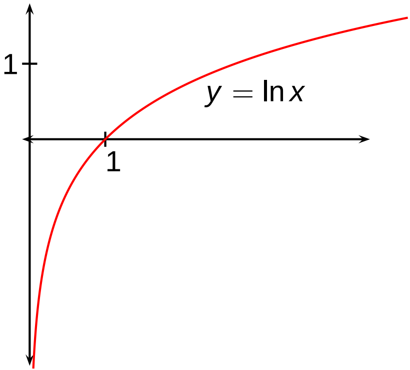

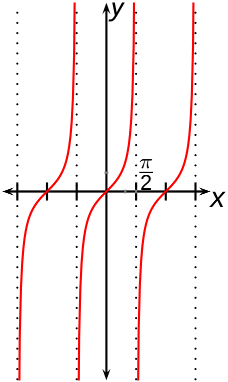

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$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to 0^-} \ln x = \text{DNE}$$

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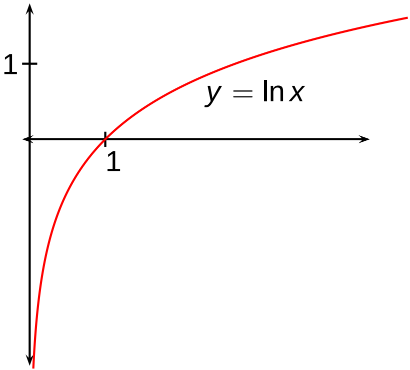

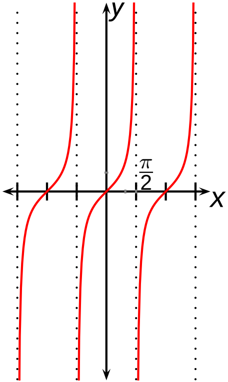

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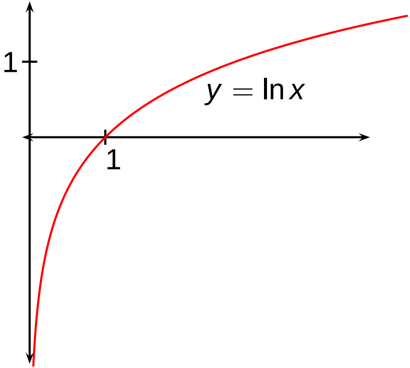

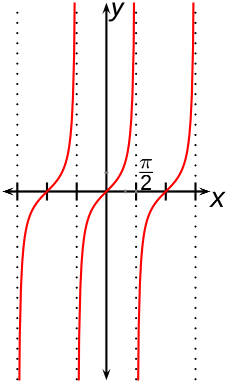

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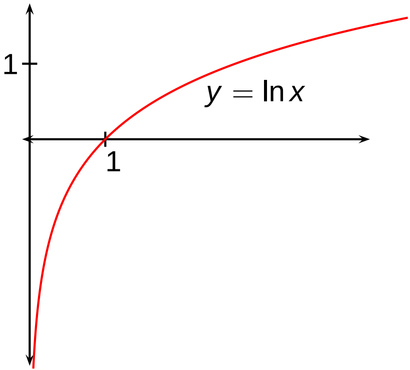

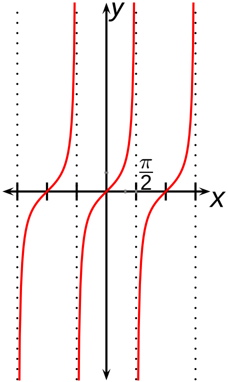

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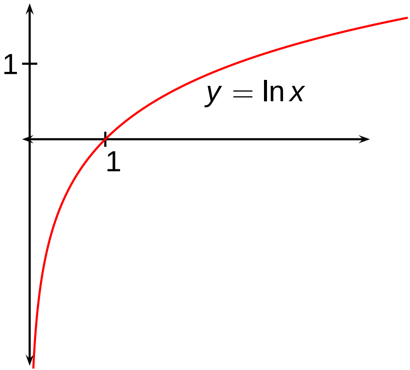

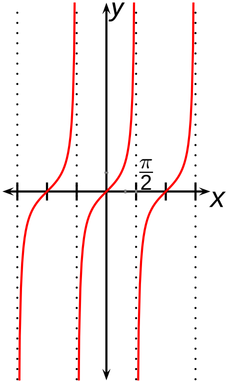

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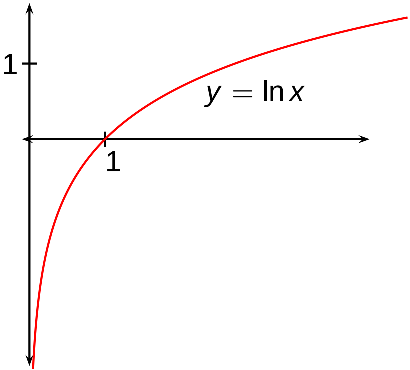

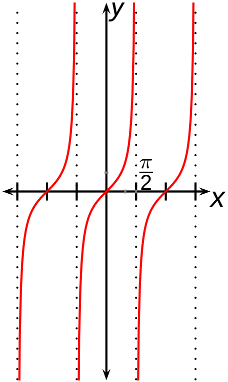

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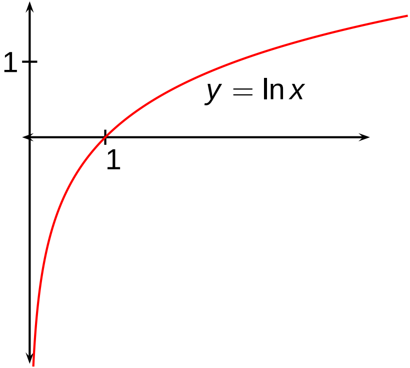

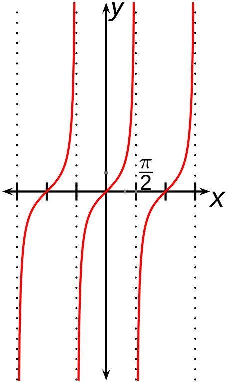

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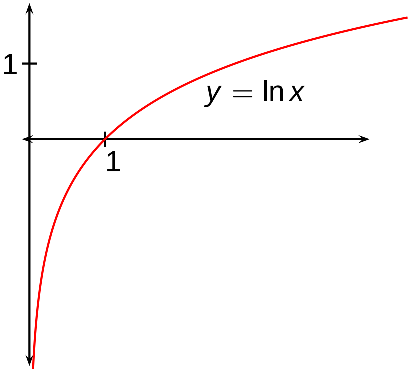

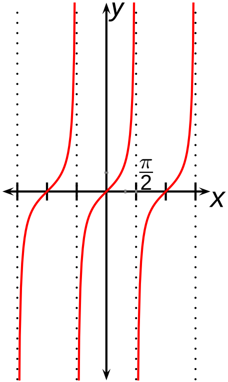

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$$\lim_{x \to 0^+} \ln x = -\infty$$

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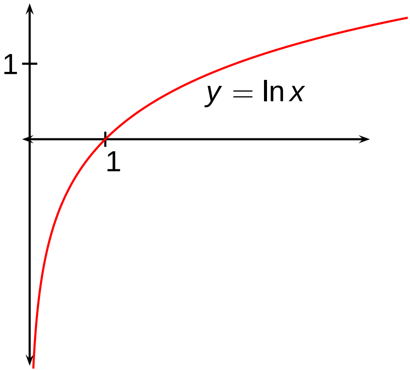

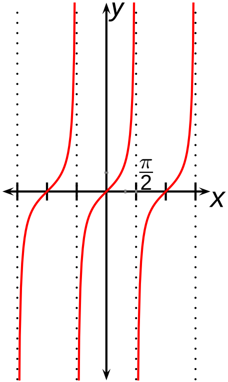

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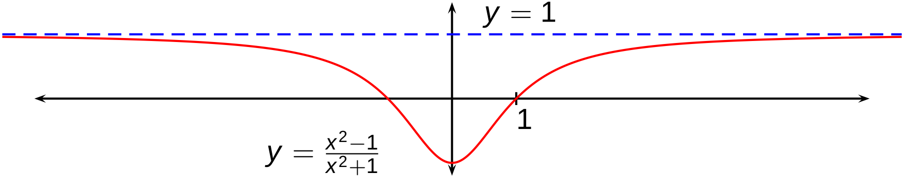
Limits at Infinity; Horizontal Asymptotes



- Consider $f(x) = \frac{x^2-1}{x^2+1}$ as x becomes large.

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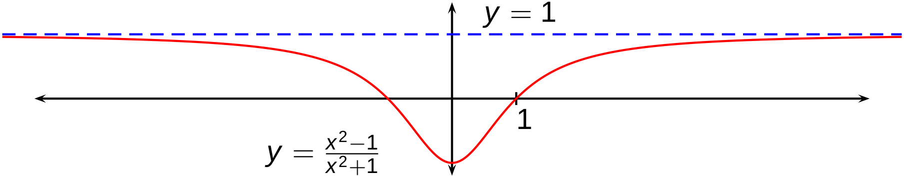

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198

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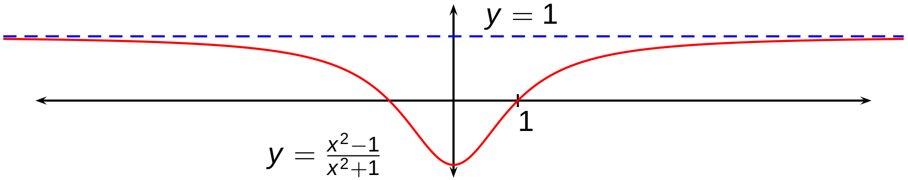

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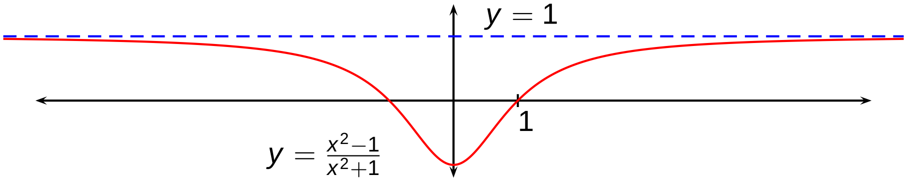

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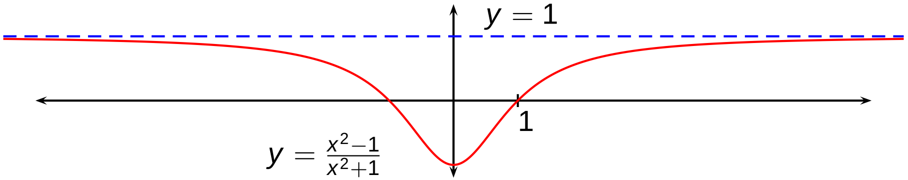

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Definition (Limit at Infinity)

Let f be a function defined on some interval (a, ∞). Then

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means that the values of f can be made arbitrarily close to L by taking x sufficiently large.

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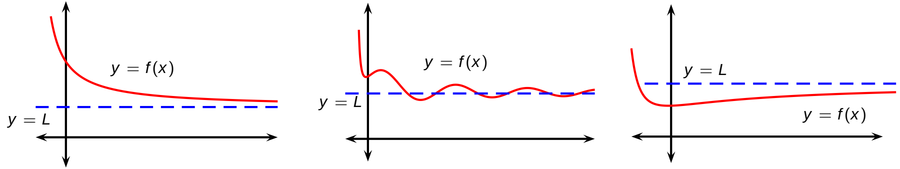

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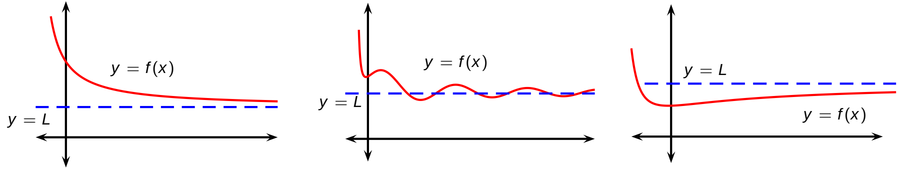

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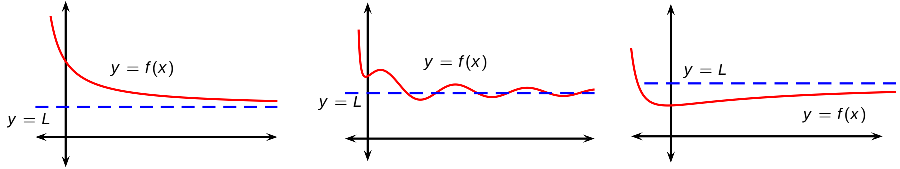

- There are many ways that this can happen.
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- ∞ is not a number.

Definition (Limit at Minus Infinity)

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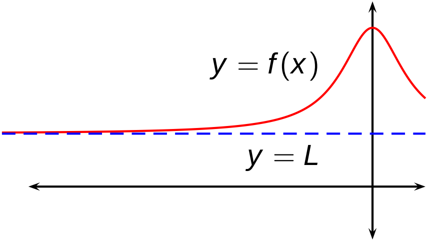
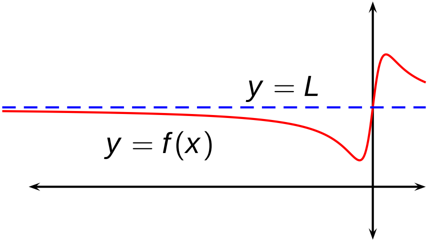
means that the values of f can be made arbitrarily close to L by taking x sufficiently large negative.





Definition (Horizontal Asymptote)

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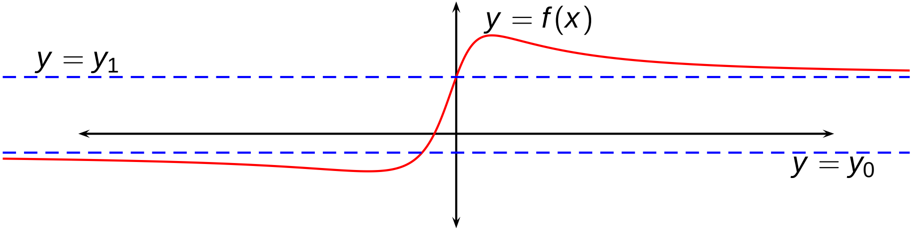
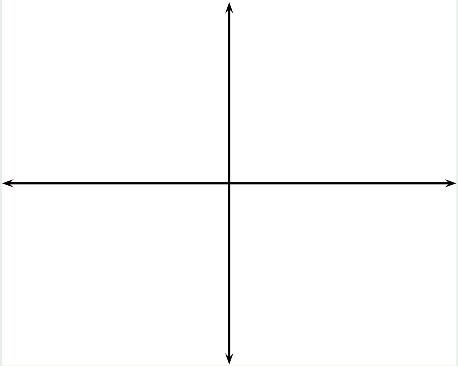
- Can a function have two horizontal asymptotes? Yes.

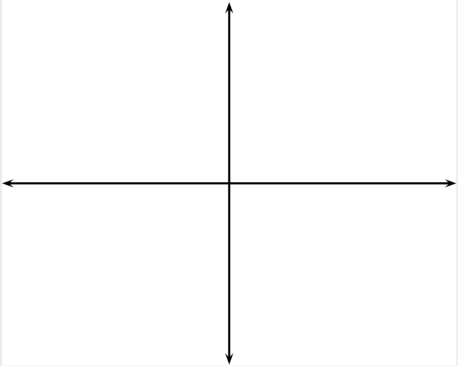


Example



Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Example



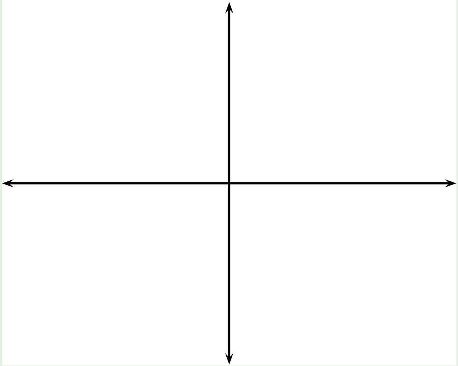
Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

- When x is large, $\frac{1}{x}$ is small.

$$\frac{1}{100} = 0.01, \qquad \frac{1}{10,000} = 0.0001$$

$$\frac{1}{1,000,000} = 0.000001$$

Example



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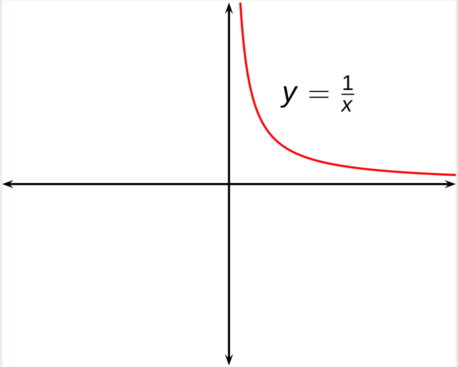

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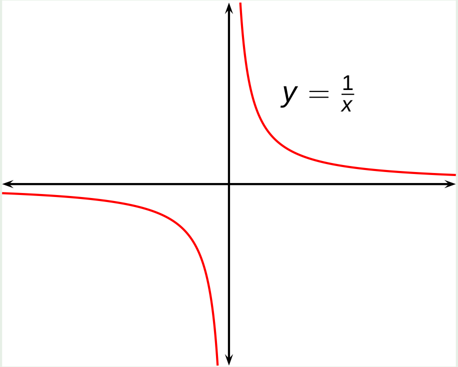

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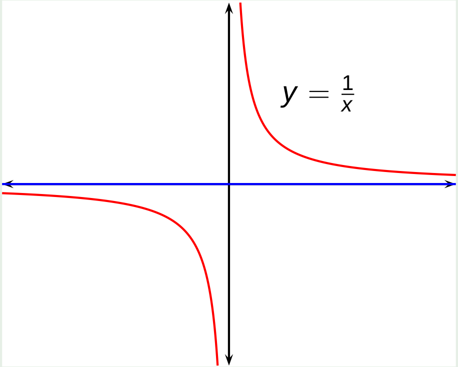

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- Therefore $\lim_{x\to\infty} \frac{1}{x} = 0$.
- Similarly, $\lim_{x\to-\infty} \frac{1}{x} = 0$.
- $y = 0$ (the x-axis) is a horizontal asymptote for the curve $y = \frac{1}{x}$.

We can generalize the previous example to other powers of x:

Theorem (Infinite Limits of $\frac{1}{x^r}$)

If $r > 0$ is a rational number, then

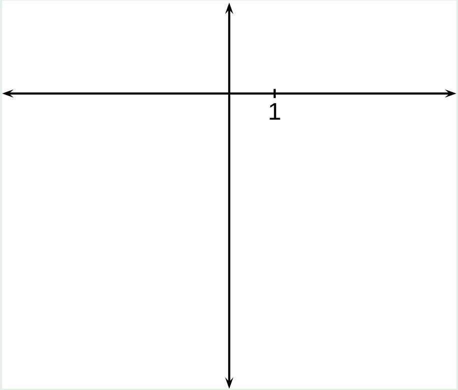
$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is an integer

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Example

Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$.



$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$$

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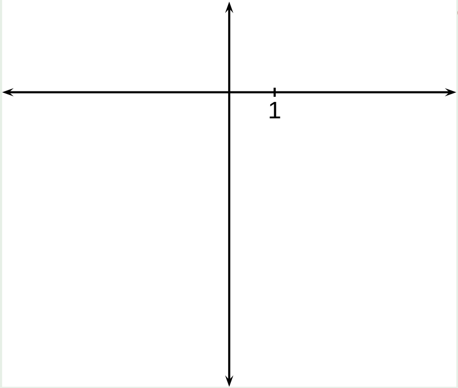

- Standard approach: divide top and bottom by the highest power of x in the denominator.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

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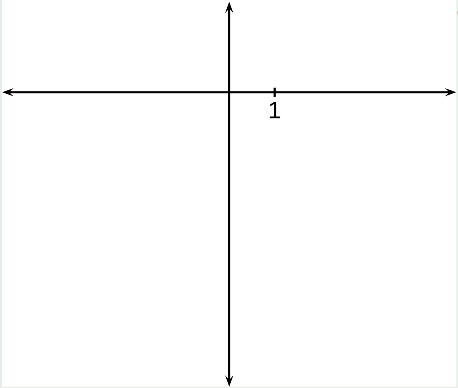

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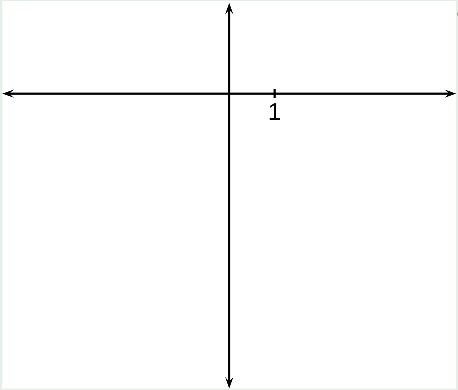

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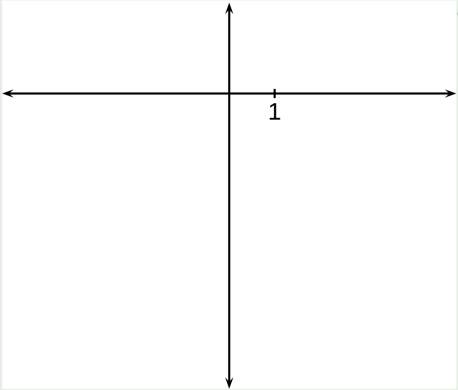

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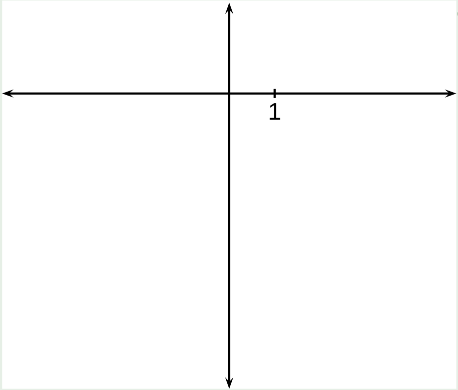

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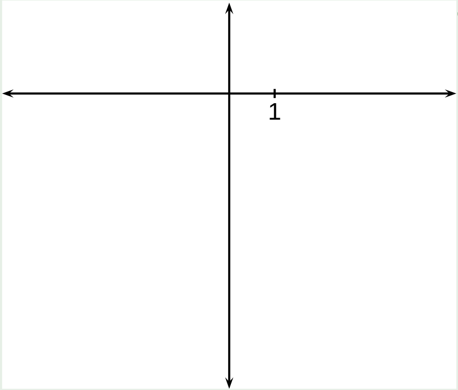

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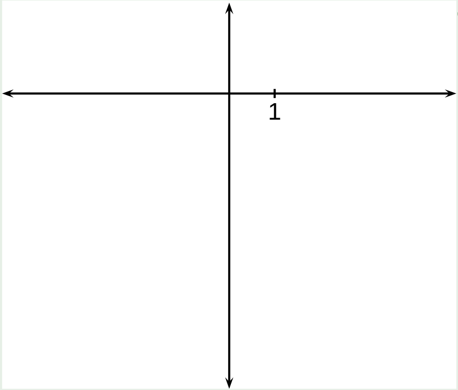

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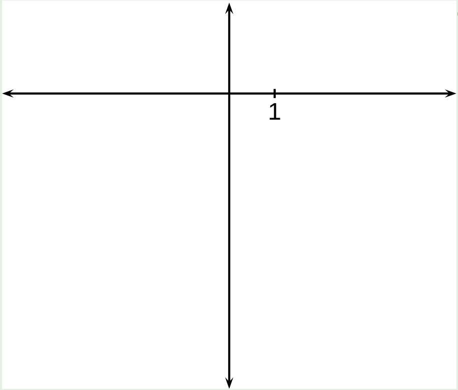

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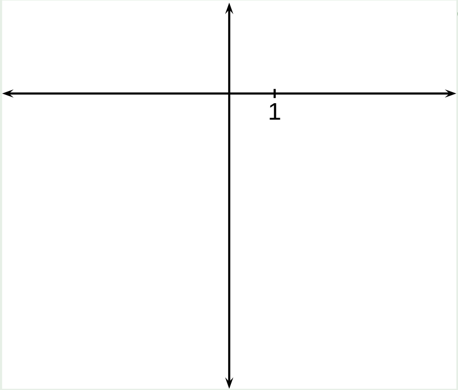

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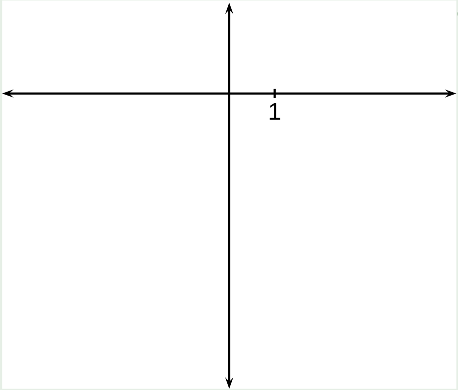

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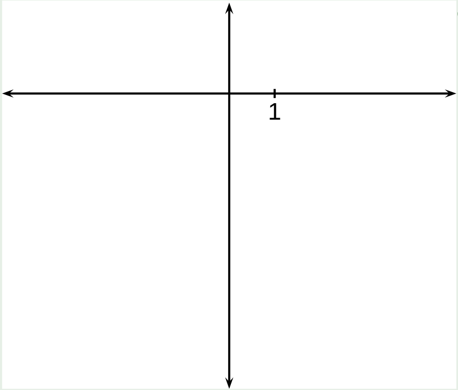

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$$\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

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$$= \frac{3 - 0 - \quad}{\quad + \quad + \quad}$$

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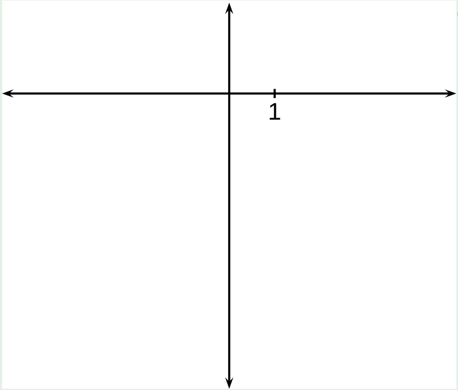

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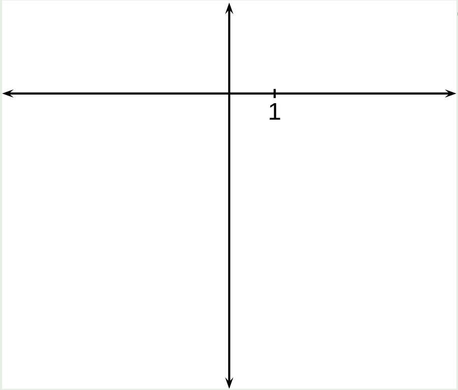

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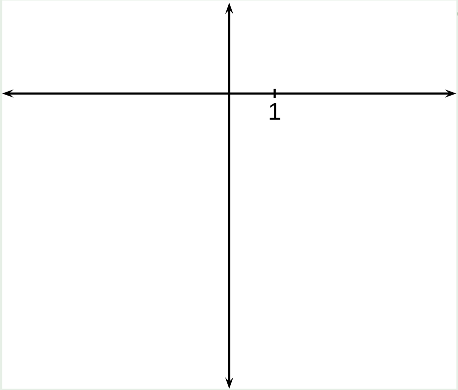

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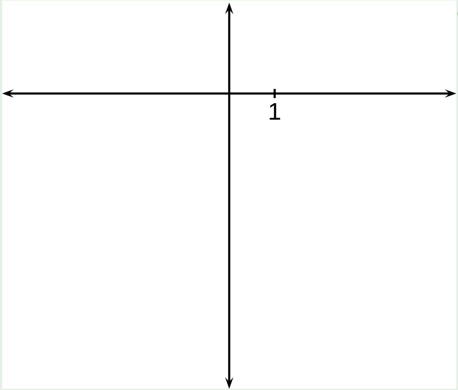

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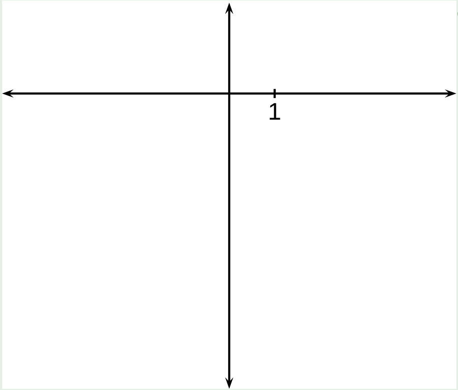

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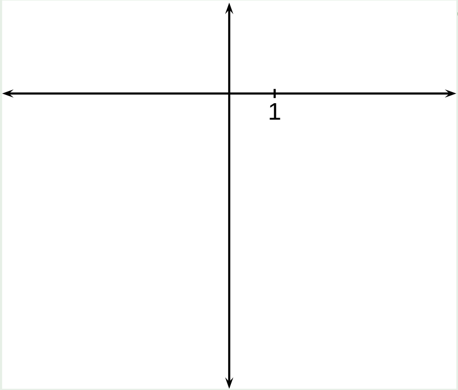

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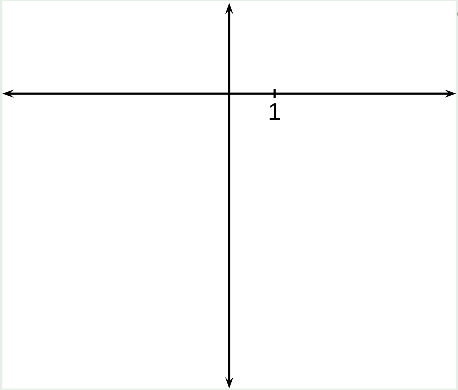

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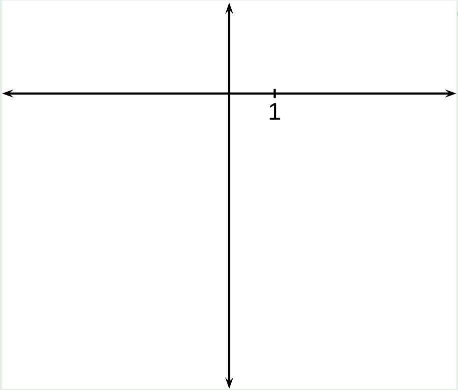

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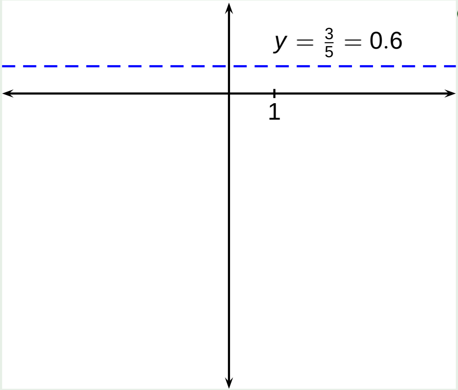

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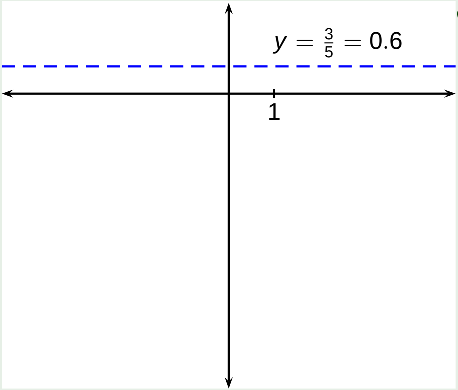

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- A similar calculation shows that the limit as $x\to-\infty$ is also $\frac{3}{5}$.

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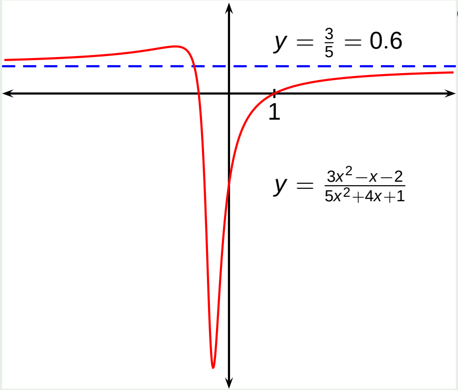

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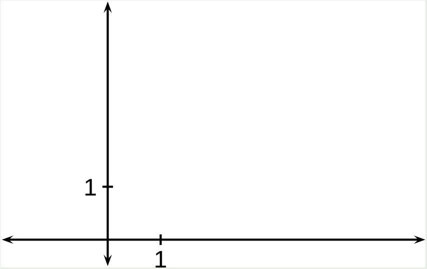
Example

Evaluate $\lim_{x\to\infty} \sqrt{x^2+1} - x$.



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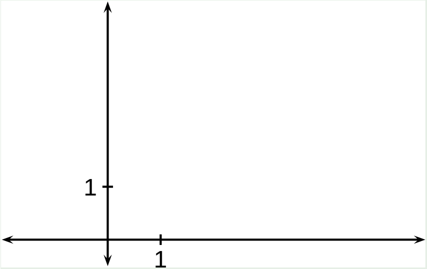

$$\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right)$$

- $\sqrt{x^2+1} \to \infty$ and $x \to \infty$ as $x \to \infty$.
- It isn't clear what happens to the difference.

Example

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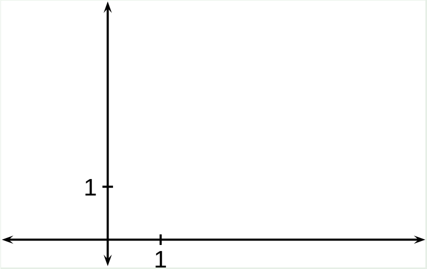

- $\sqrt{x^2+1} \to \infty$ and $x \to \infty$ as $x \to \infty$.
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- Standard approach: multiply top and bottom by conjugate radical.

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$$

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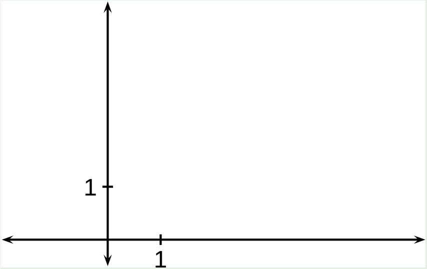

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$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

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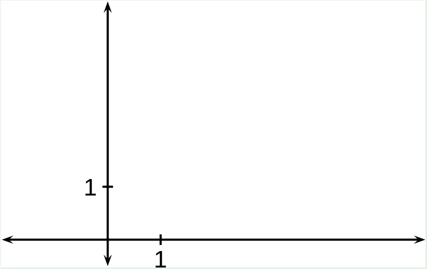

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$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

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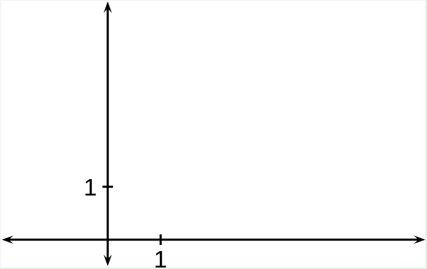

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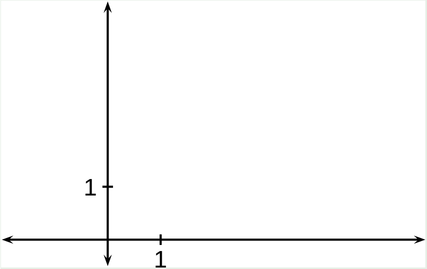

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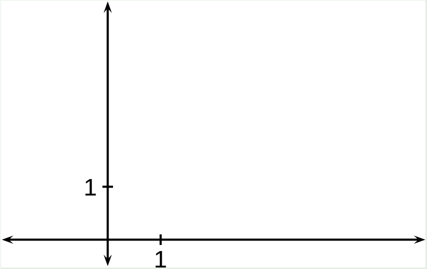

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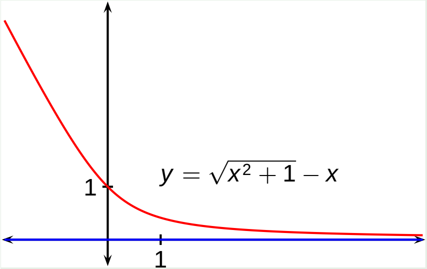

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$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}$$
$$= \frac{0}{\sqrt{1+0}+1} = 0$$

Infinite Limits at Infinity

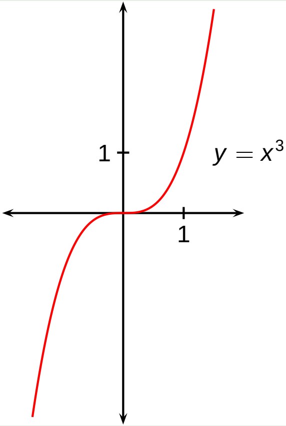
We write

$$\lim_{x\to\infty} f(x) = \infty$$

to mean that $f(x)$ becomes large as x becomes large. We attach similar meaning to

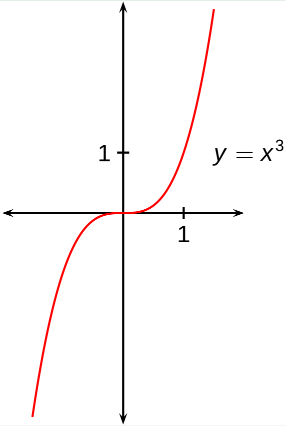
$$\lim_{x\to\infty} f(x) = -\infty, \qquad \lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} = -\infty$$

Example



Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

Example

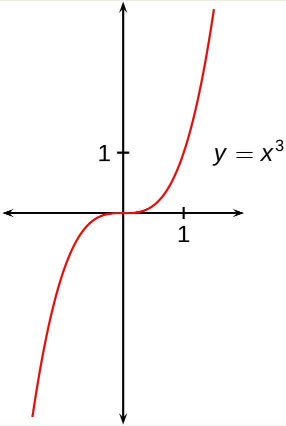


Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- When x is large, so is x^3.

$$10^3 = 1000, \qquad 100^3 = 1,000,000,$$
$$1000^3 = 1,000,000,000$$

Example



Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- When x is large, so is x^3.
- By taking x large enough, we can make x^3 as large as we like.

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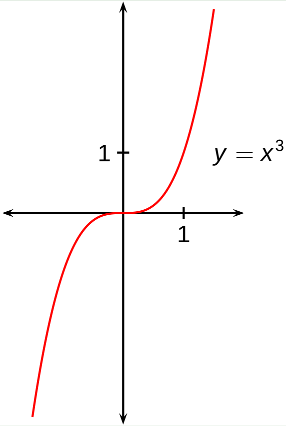

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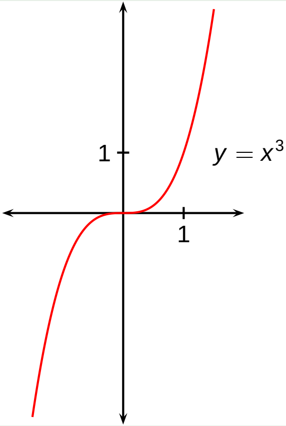
Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- When x is large, so is x^3.
- By taking x large enough, we can make x^3 as large as we like.
- Therefore $\lim_{x\to\infty} x^3 = \infty$.

Example



$10^3 = 1000, \qquad 100^3 = 1,000,000,$
$1000^3 = 1,000,000,000$

Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- When x is large, so is x^3.
- By taking x large enough, we can make x^3 as large as we like.
- Therefore $\lim_{x\to\infty} x^3 = \infty$.
- Similarly, $\lim_{x\to-\infty} x^3 = -\infty$.

Example

Find $\lim_{x\to\infty}(x^2 - x)$.

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- WRONG: $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty = 0$.

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- We can't use the limit laws in this way with ∞ because ∞ isn't a number.

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- Instead: $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$.

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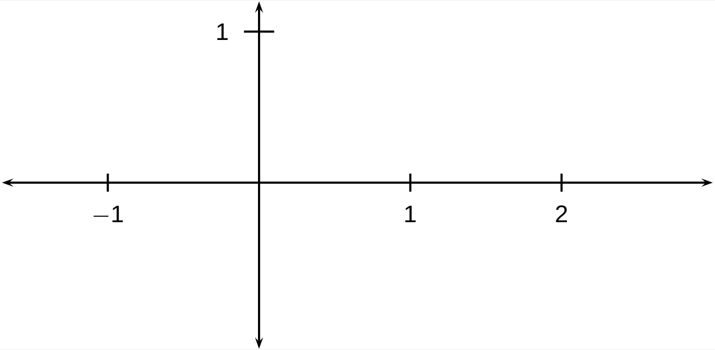
- We can't use the limit laws in this way with ∞ because ∞ isn't a number.
- Instead: $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x - 1) = \infty.$
- This is because x and $x - 1$ both become arbitrarily large as $x \to \infty$.

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



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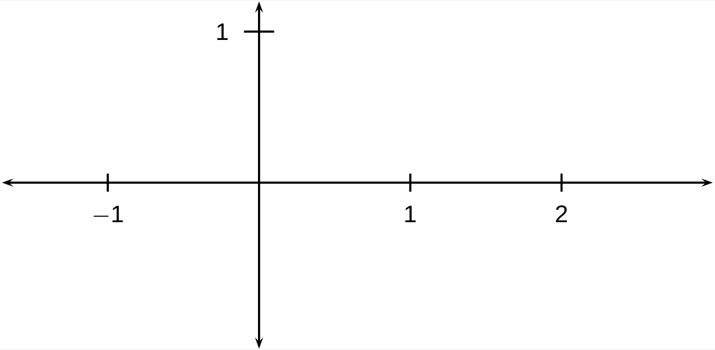

$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

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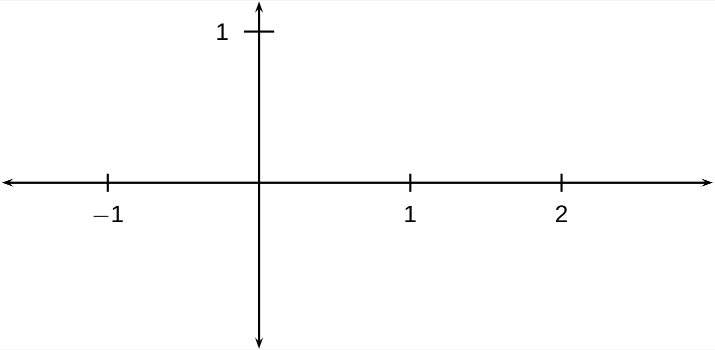

$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

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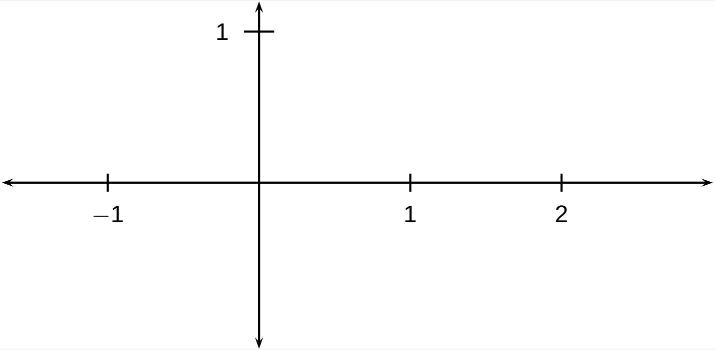


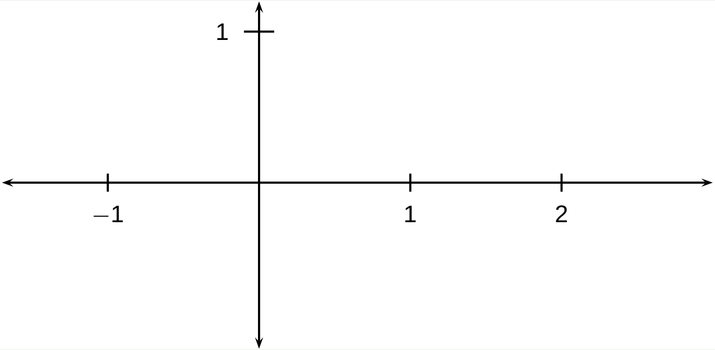
$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

$+$

$$\lim_{x\to-\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



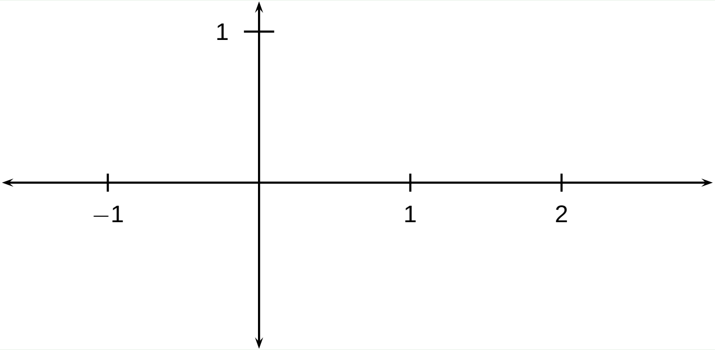
$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4 \color{red}{(x+1)^3}(x-1) \quad =$$

$$+$$

$$\lim_{x\to-\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4{\color{red}(x+1)^3}(x-1) \quad =$$

$$\qquad + \qquad\qquad {\color{red}+}$$

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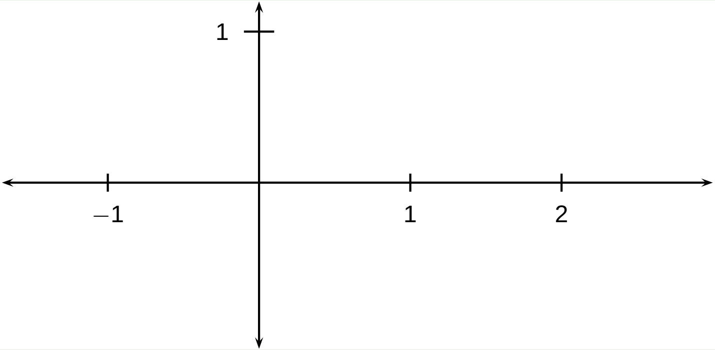
Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3{\color{red}(x-1)} =$$

$$+ \qquad +$$

$$\lim_{x\to-\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) =$$

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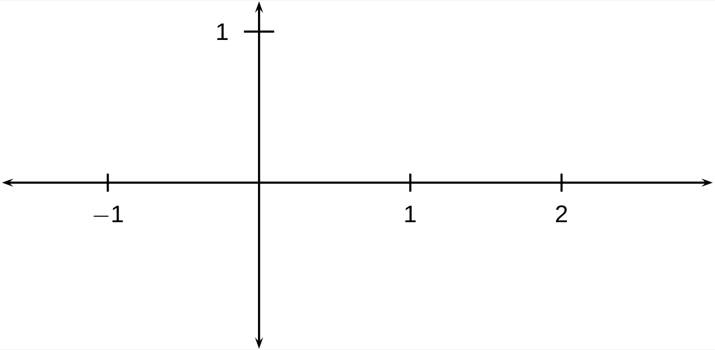
Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) \quad =$$

$$+ \qquad + \qquad +$$

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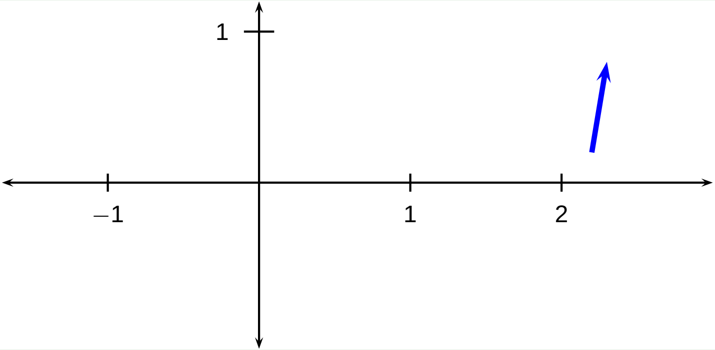
Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



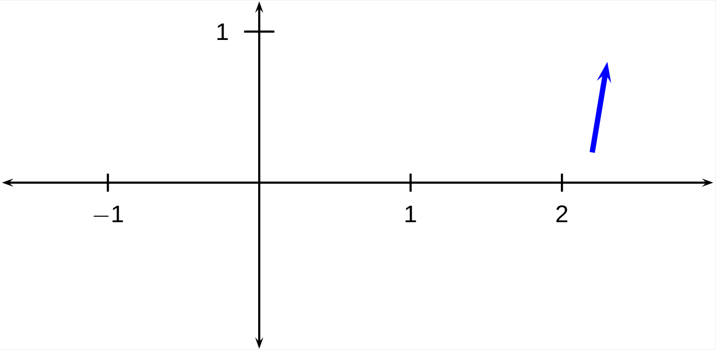
$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$$\quad + \quad + \quad +$$

$$\lim_{x\to-\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) =$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



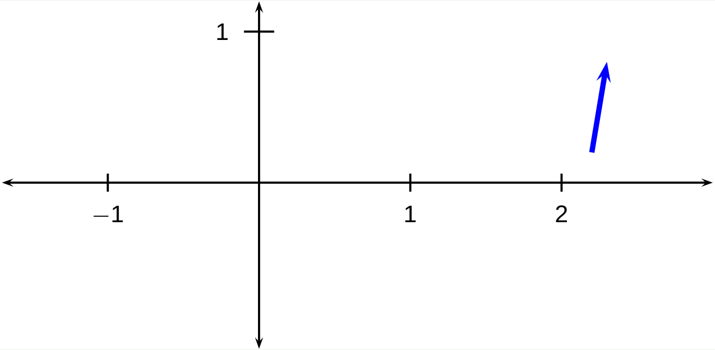
$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$$+ \quad + \quad +$$

$$\lim_{x\to-\infty} \frac{1}{24}\color{red}{(x-2)}^{\color{red}{4}}(x+1)^3(x-1) =$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$$\qquad + \qquad + \qquad +$$

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$$+$$

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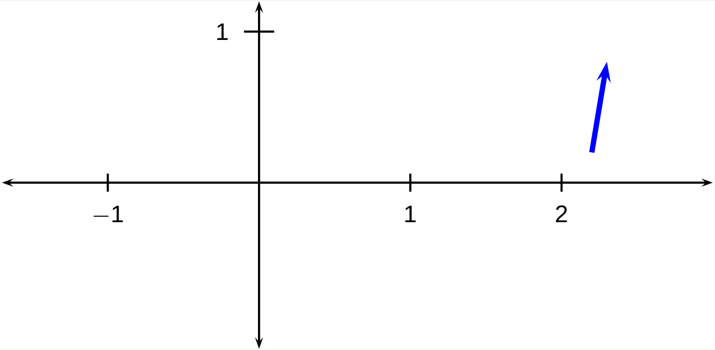
Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



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$$+ \quad + \quad +$$

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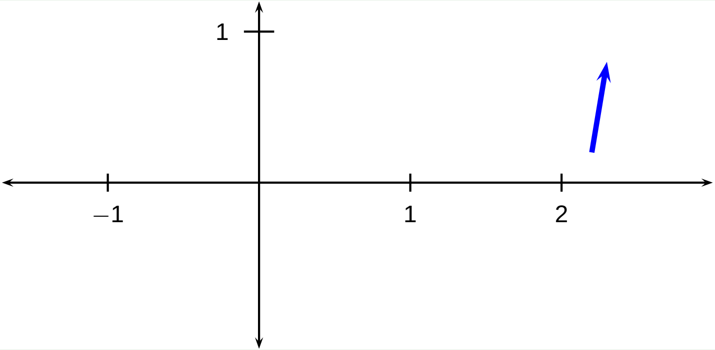

$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$+ \quad + \quad +$

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$+ \quad -$

Example

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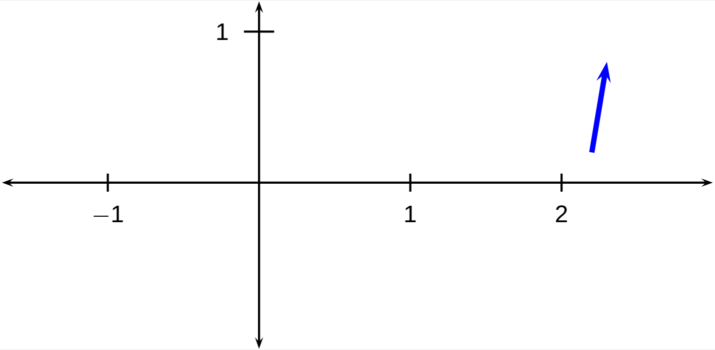
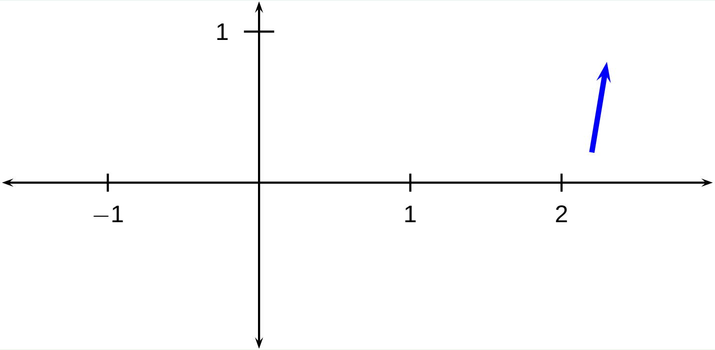


$$\lim_{x\to\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$
$$+ \quad + \quad +$$

$$\lim_{x\to-\infty} \frac{1}{24}(x-2)^4(x+1)^3{\color{red}(x-1)} =$$
$$+ \quad -$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.

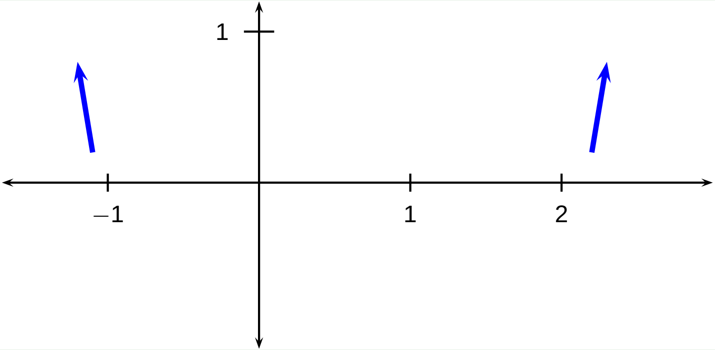


$$\lim_{x\to\infty} \underbrace{\tfrac{1}{24}}_{}\underbrace{(x-2)^4}_{+}\underbrace{(x+1)^3}_{+}\underbrace{(x-1)}_{+} = \infty$$

$$\lim_{x\to-\infty} \tfrac{1}{24}\underbrace{(x-2)^4}_{+}\underbrace{(x+1)^3}_{-}\underbrace{\color{red}{(x-1)}}_{\color{red}{-}} =$$

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of $y = \frac{1}{24}(x-2)^4(x+1)^3(x-1)$.



$$\lim_{x \to \infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$$+ \quad + \quad +$$

$$\lim_{x \to -\infty} \frac{1}{24}(x-2)^4(x+1)^3(x-1) = \infty$$

$$+ \quad - \quad -$$

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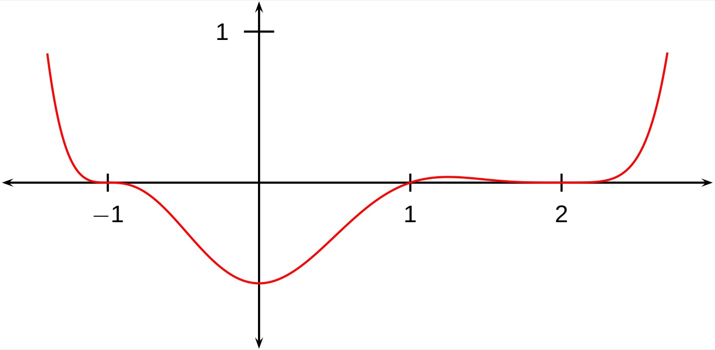

$$\lim_{x\to\infty} \underbrace{\tfrac{1}{24}}_{}\underset{+}{(x-2)^4}\underset{+}{(x+1)^3}\underset{+}{(x-1)} = \infty$$

$$\lim_{x\to-\infty} \tfrac{1}{24}\underset{+}{(x-2)^4}\underset{-}{(x+1)^3}\underset{-}{(x-1)} = \infty$$