Math 140

Lecture 9

Greg Maloney

with modifications by T. Milev

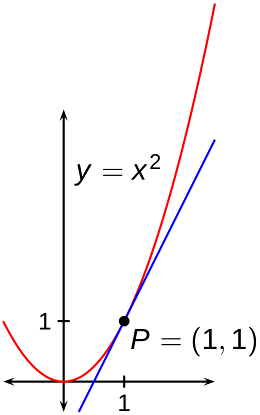
University of Massachusetts Boston

February 28, 2013

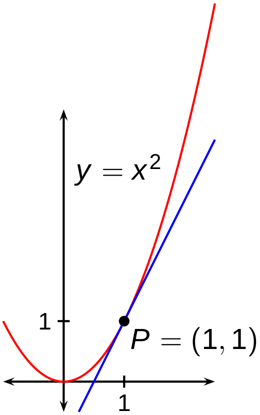
Outline

Outline

Tangents

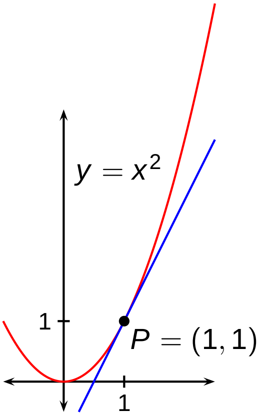


Tangents

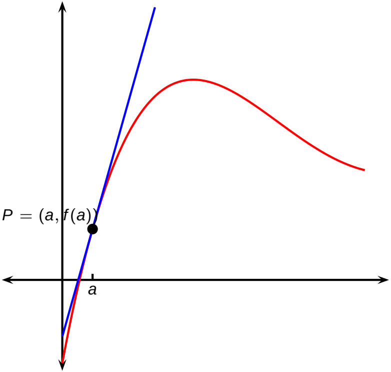


- Recall that in section (2.1) we tried to find the tangent line to the curve $y = x^2$ at the point $P = (1, 1)$.

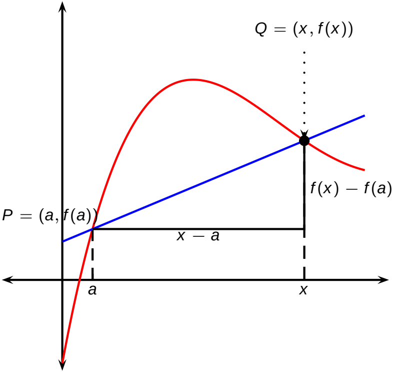
Tangents



- Recall that in section (2.1) we tried to find the tangent line to the curve $y = x^2$ at the point $P = (1, 1)$.
- This problem motivated us to study limits.



- How to find the tangent line to the curve $y = f(x)$ at $P = (a, f(a))$?



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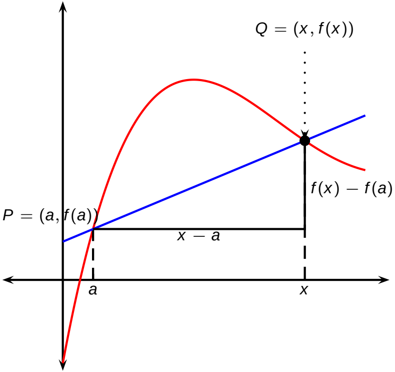

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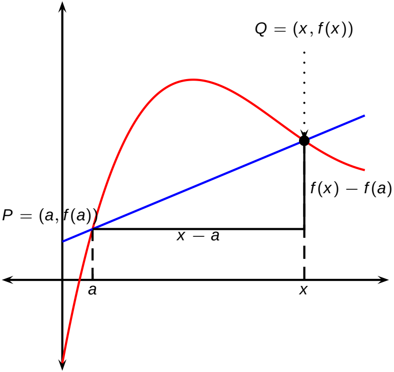

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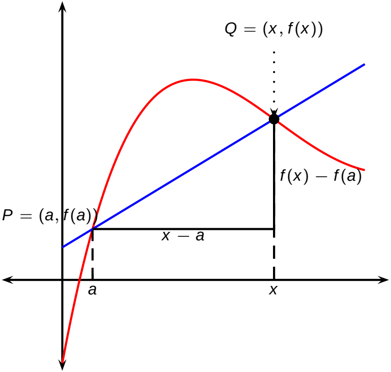

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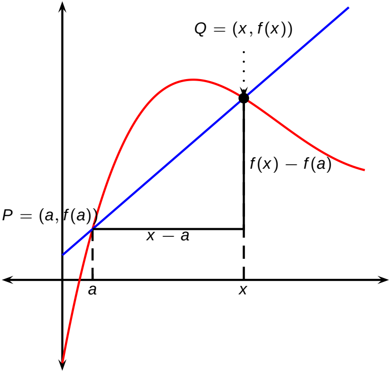

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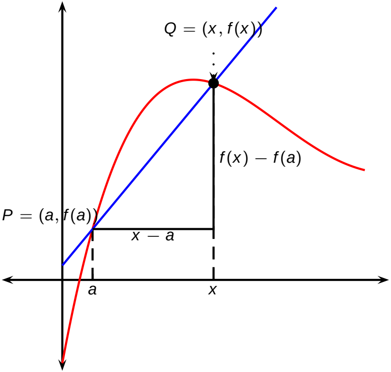

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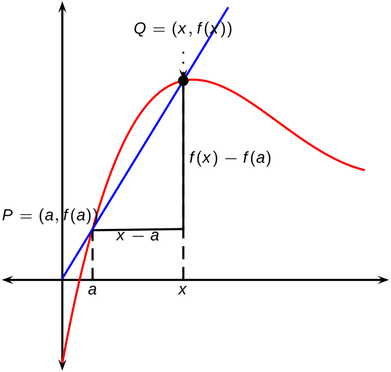

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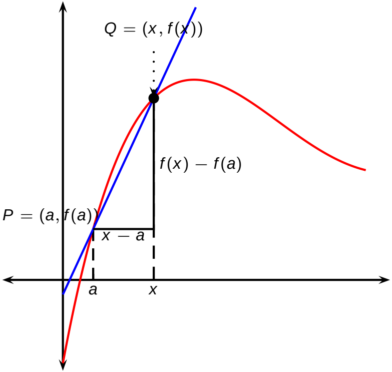

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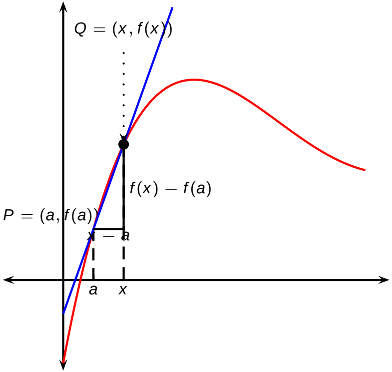

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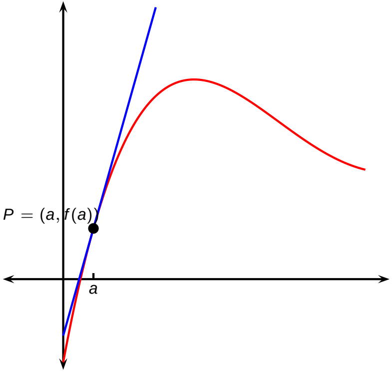

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Definition (Tangent Line)

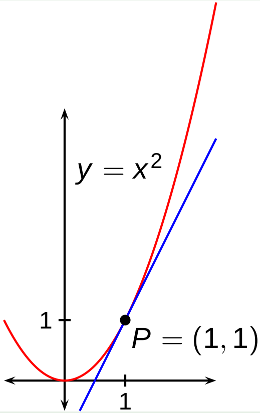
The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Example

Find an equation for the tangent line to the parabola $y = x^2$ at the point $P = (1, 1)$.



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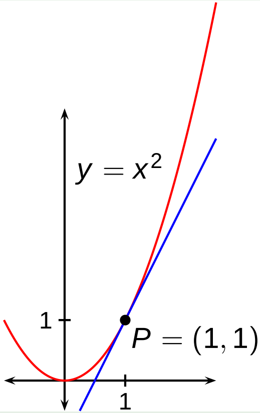

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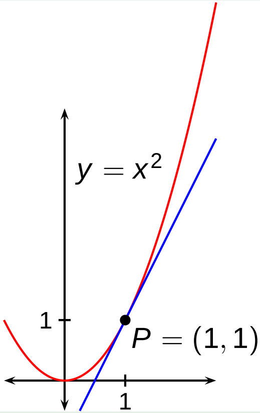

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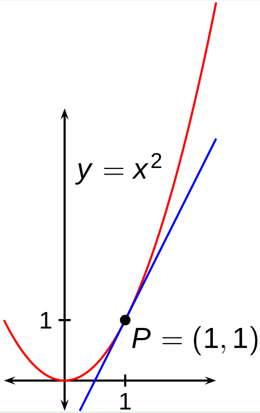

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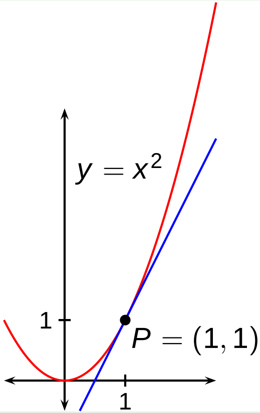

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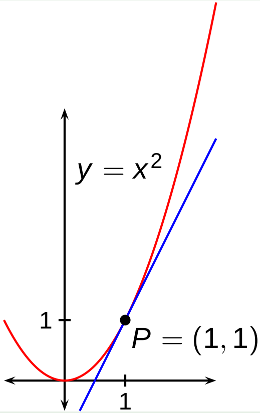

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$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x - 1} \\ &= \lim_{x\to 1} (x + 1) \end{aligned}$$

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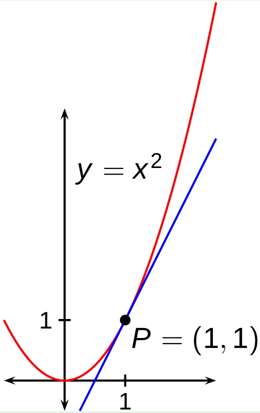

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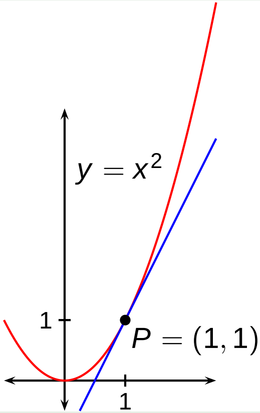

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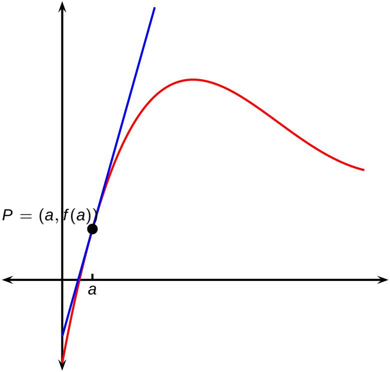
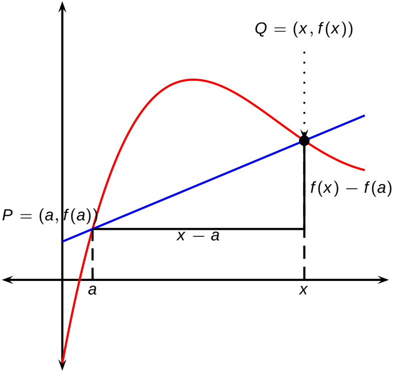
Point-slope form: $y - 1 = 2(x - 1)$, or $y = 2x - 1$.



- There is another expression for the slope of the tangent line.



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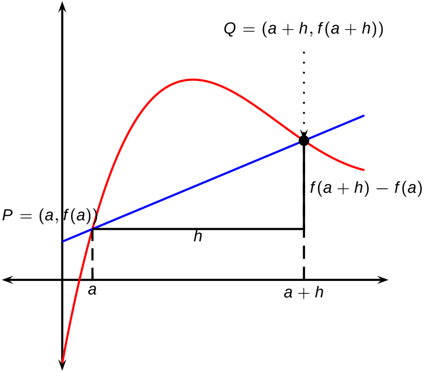

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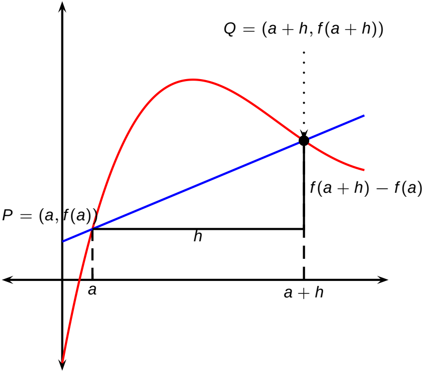

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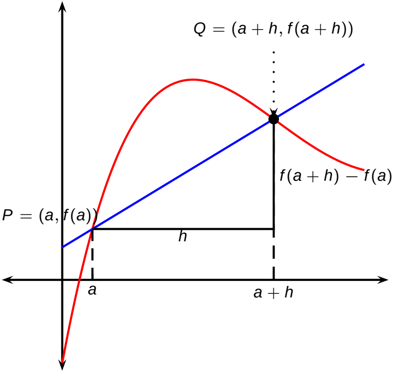

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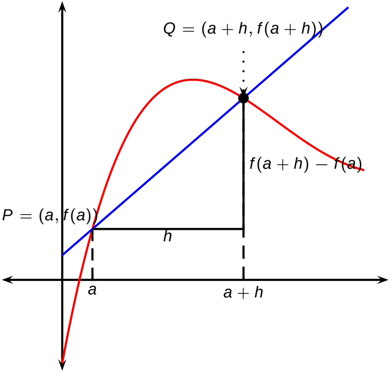

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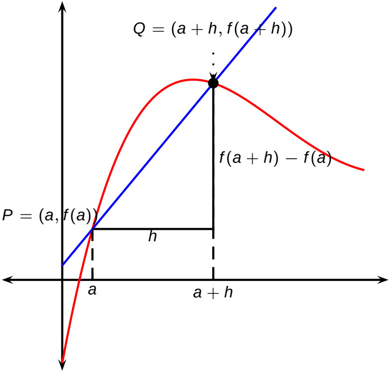

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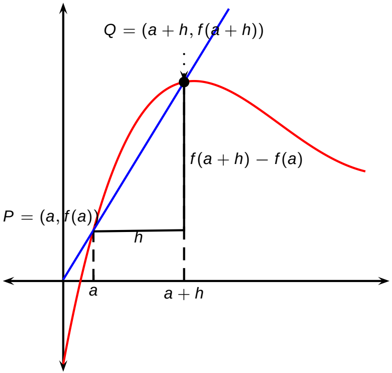

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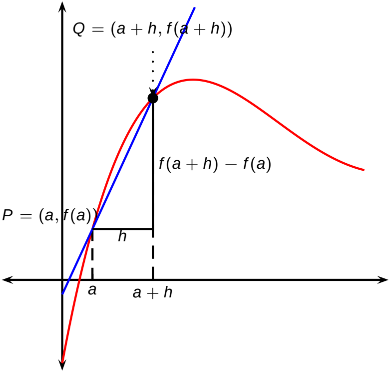

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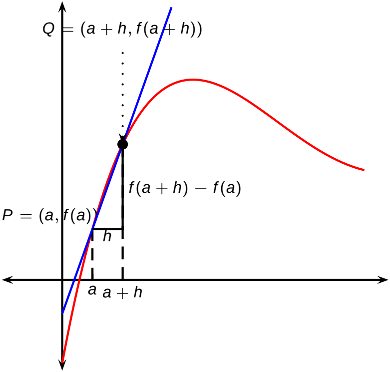

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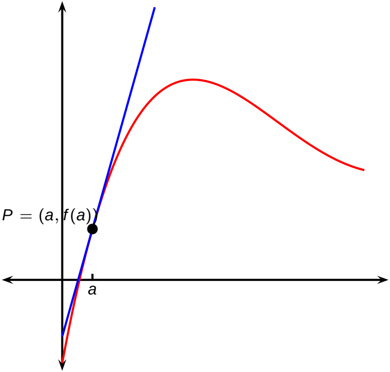

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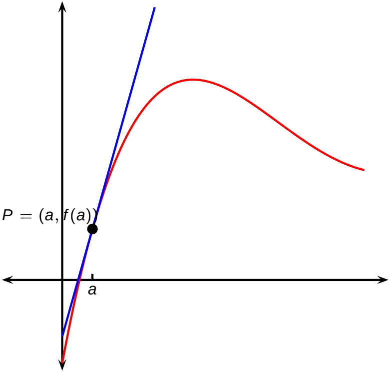

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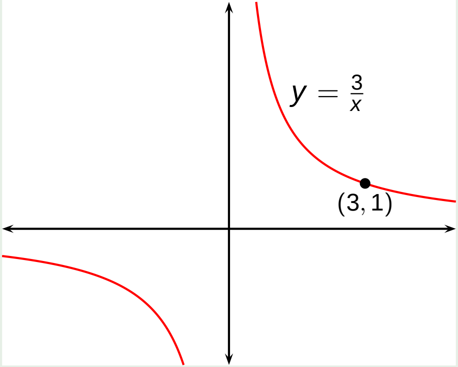
Alternative formula:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example

Find an equation for the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.



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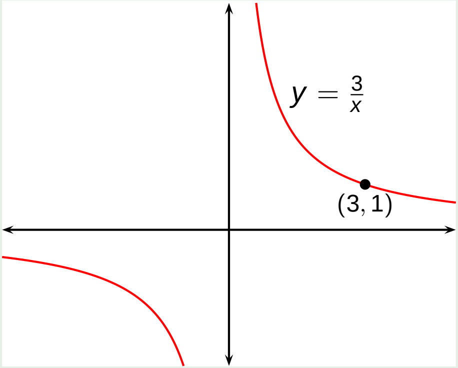

Here $a = 3$ and $f(x) = 3/x$.

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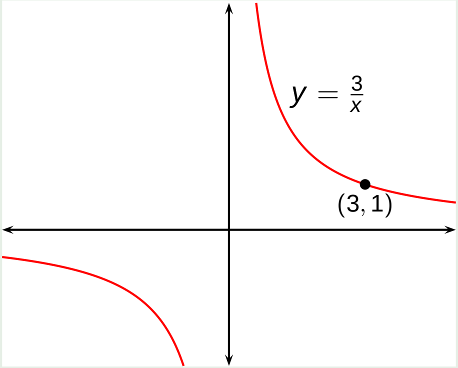

Here $a = 3$ and $f(x) = 3/x$.

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

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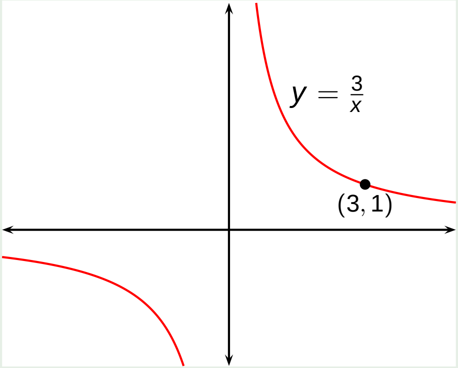

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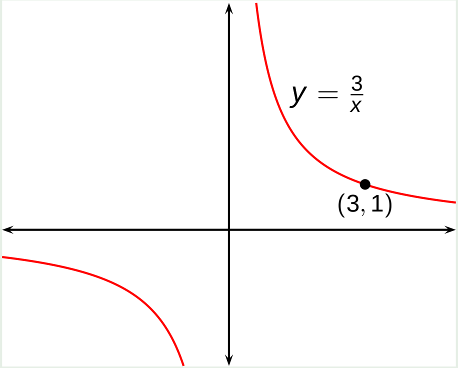

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$$m = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$

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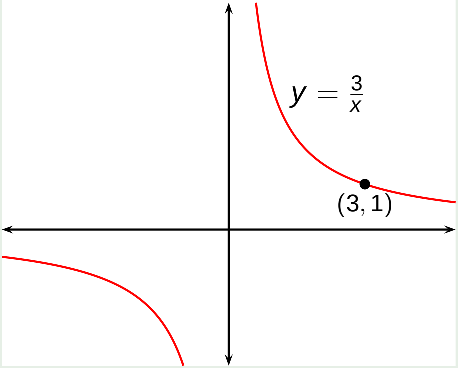

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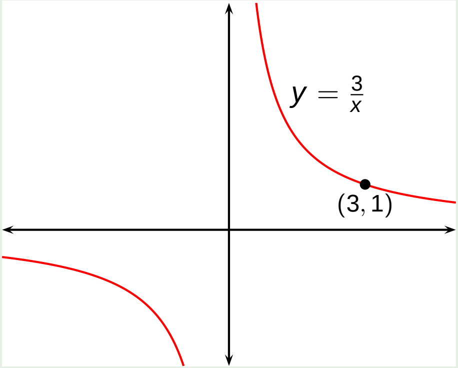

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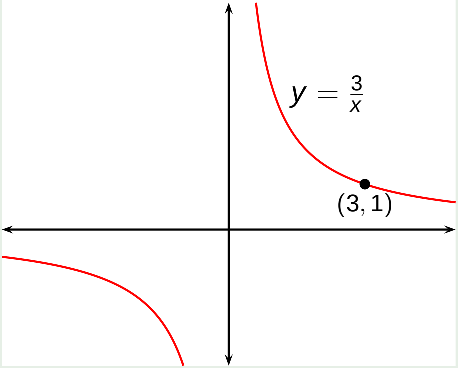
Example

Find an equation for the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.



Here $a = 3$ and $f(x) = 3/x$.

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\begin{aligned}
m &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\
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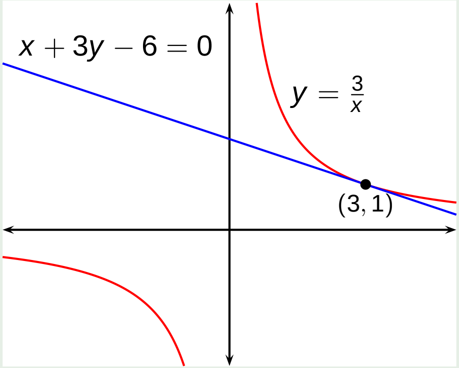

Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$, or $x + 3y - 6 = 0$.

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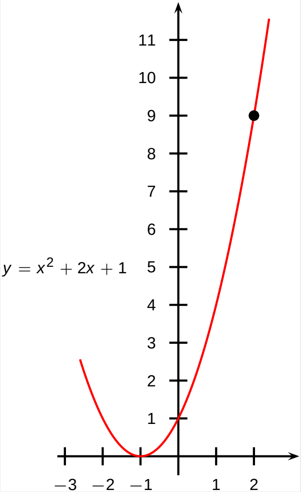
Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



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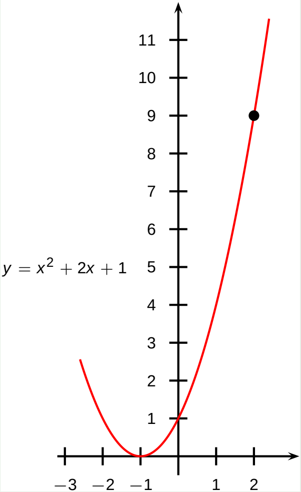

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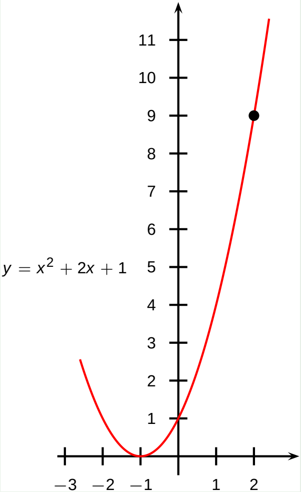

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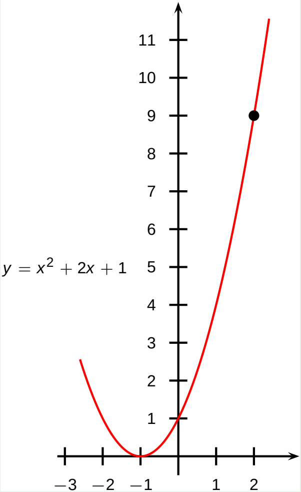

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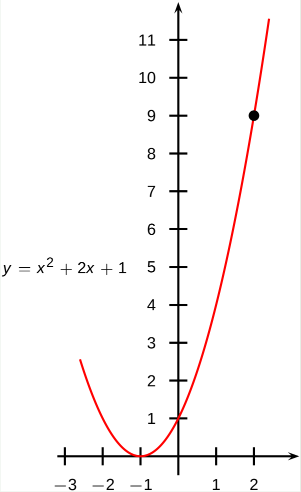

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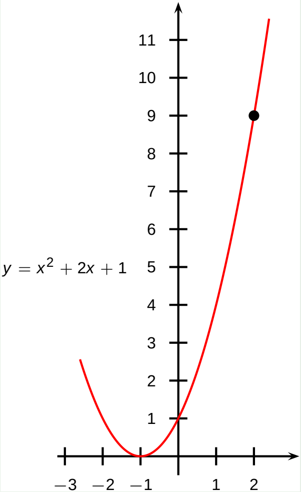

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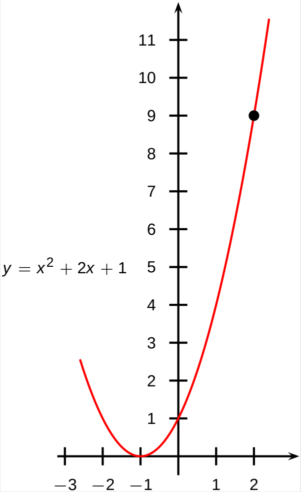

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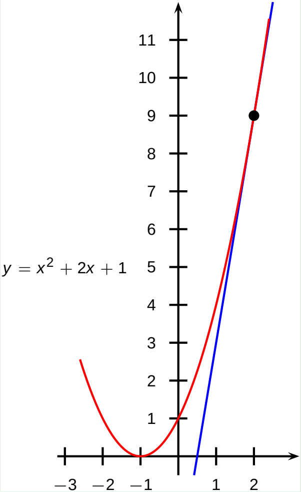

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The tangent line: $y = 6x - 3$.

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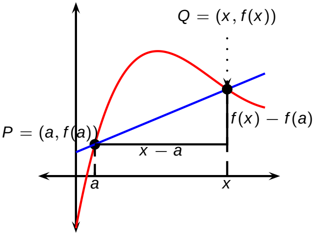

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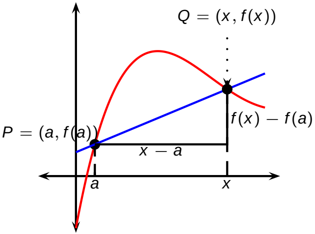

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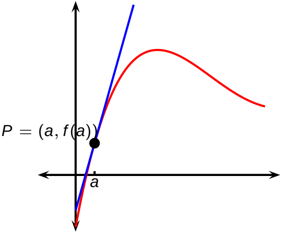

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Therefore the velocity after 5s is $v(5) = 9.8(5) = 49$m/s.

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Limits of the type

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Alternative formula:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

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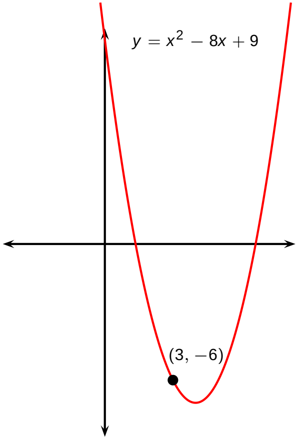
Find an equation for the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $P = (3, -6)$.



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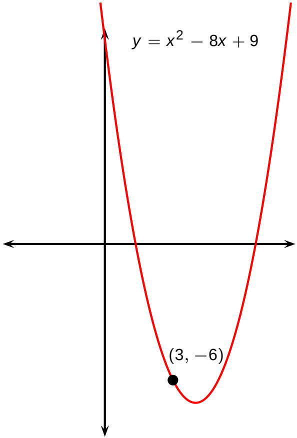

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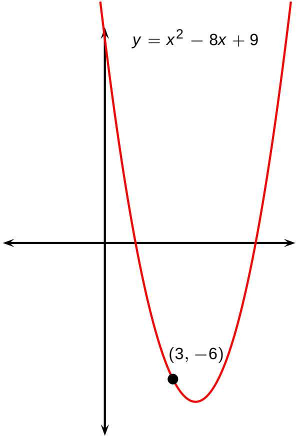

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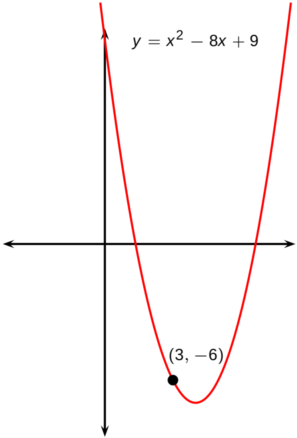

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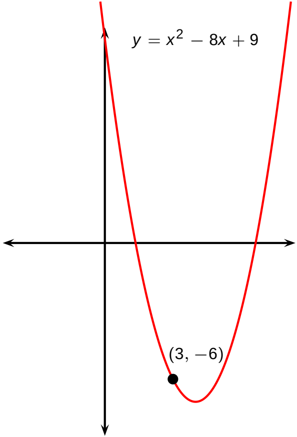

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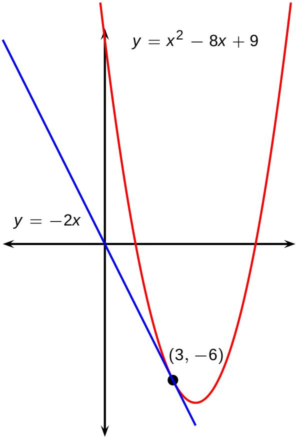

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- Slope y-intercept form: $y = -2x$.

The Derivative as a Function

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

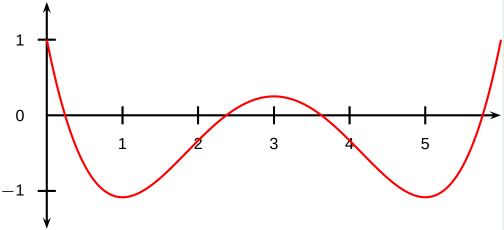
Now we change our point of view by letting the number a vary. If we replace the number a with the variable x, we get

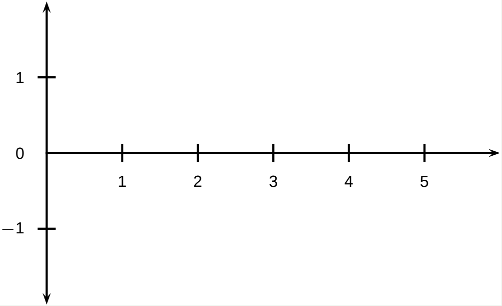
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We regard f' as a new function, called the derivative of f.
The domain of f' is $\{x | f'(x) \text{ exists } \}$. It may be smaller than the domain of f.

Example

The graph of a function f appears below. Use it to sketch the graph of the derivative f'.





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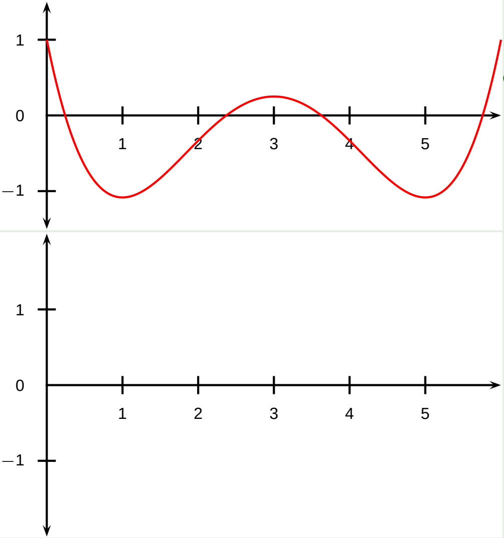

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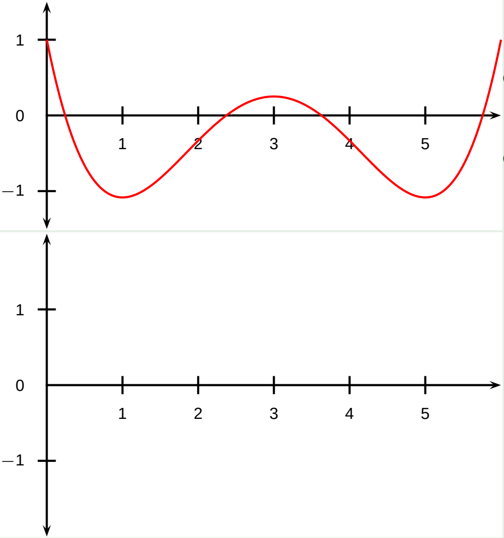

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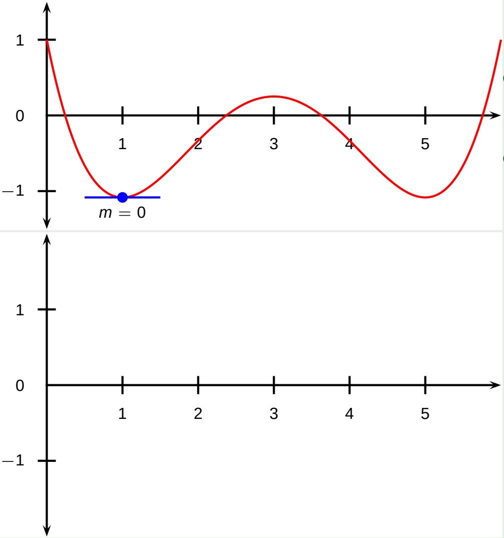

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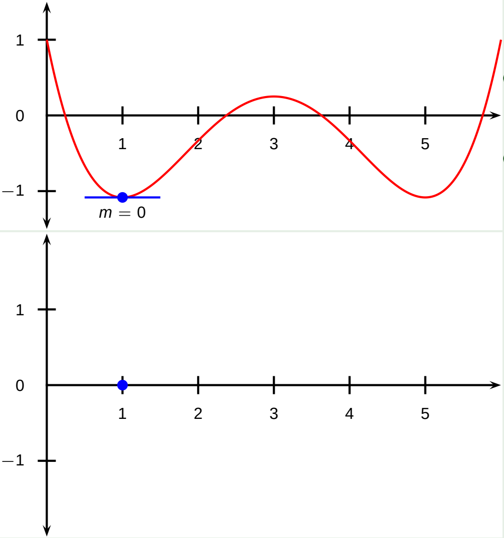

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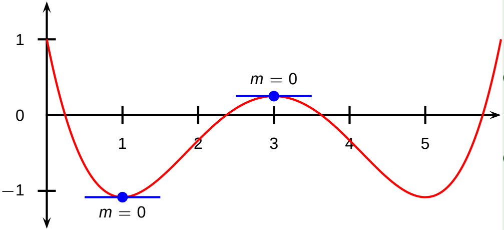

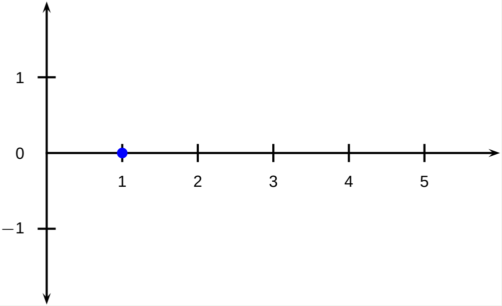

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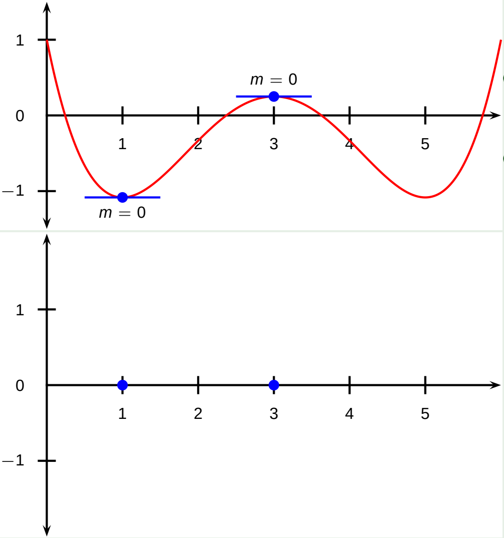

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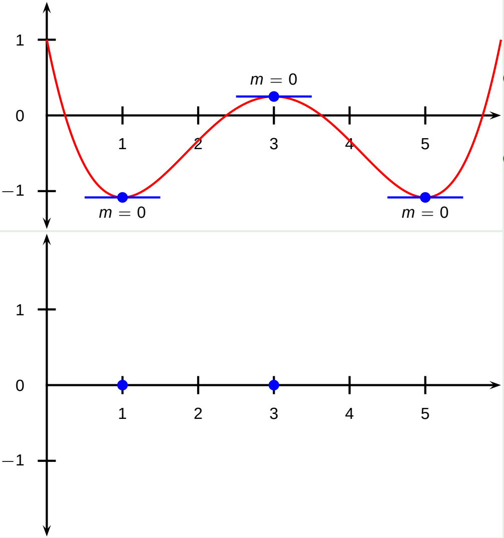

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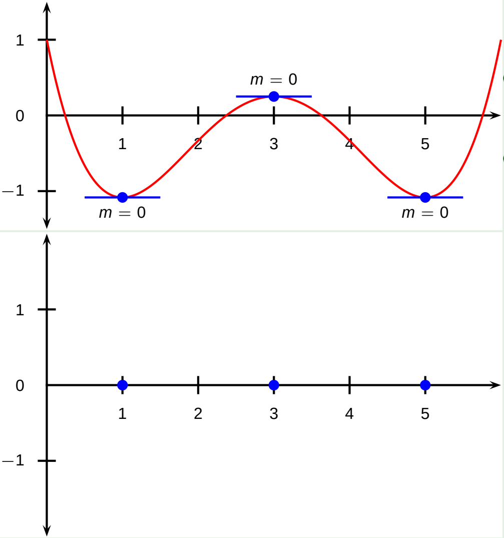

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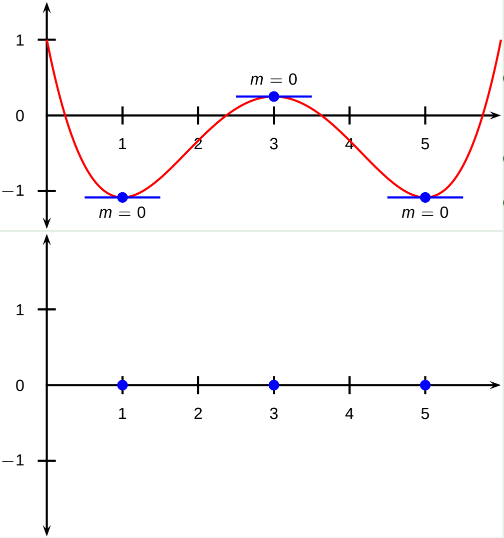

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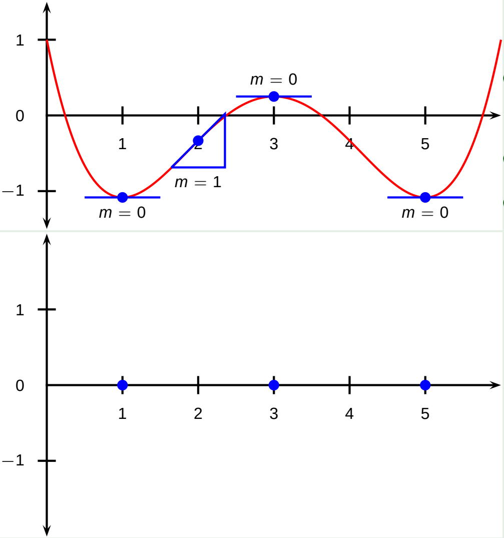

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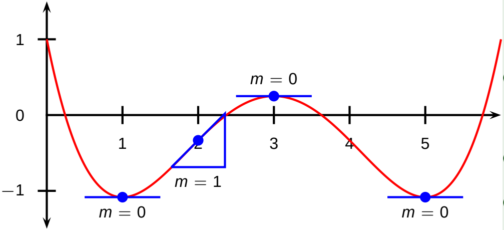

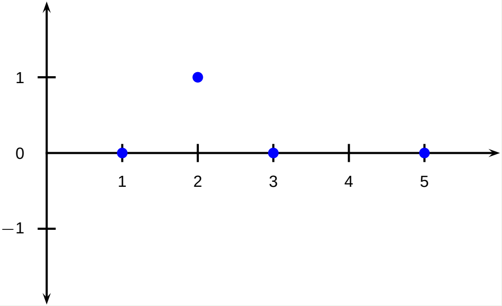

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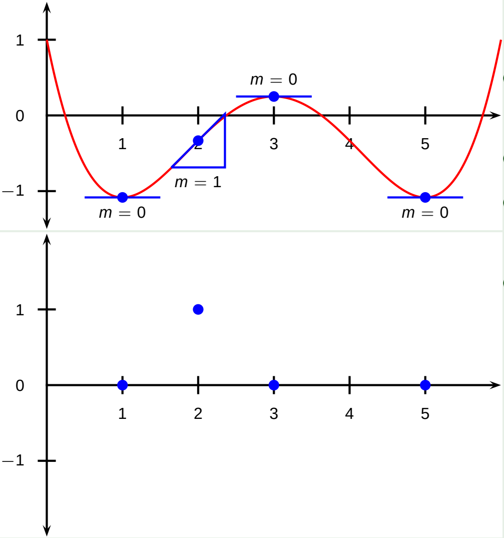

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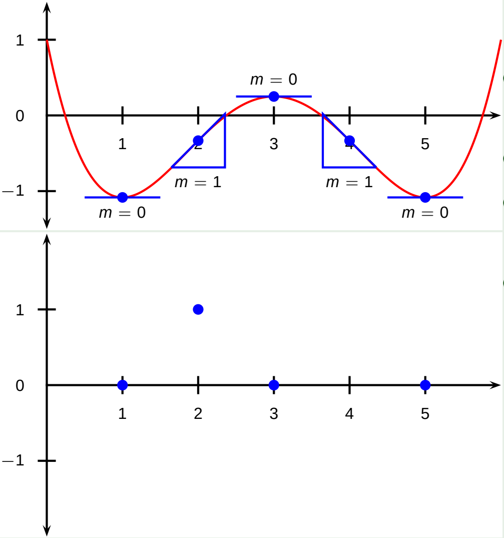

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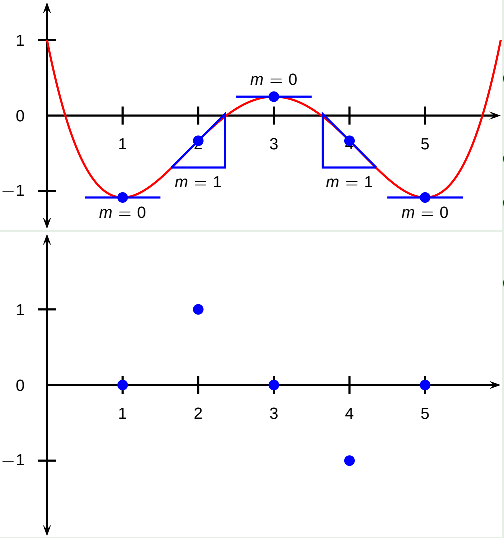

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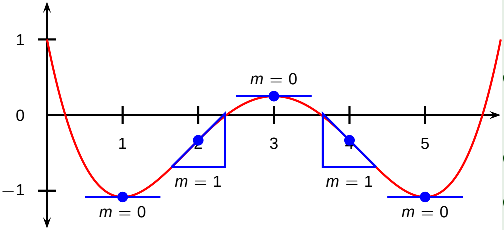

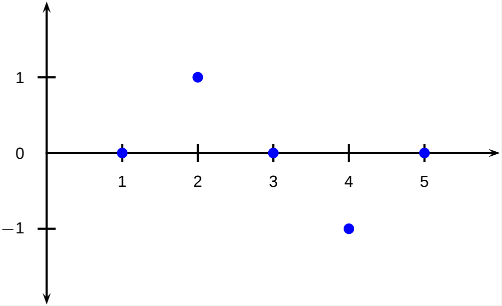

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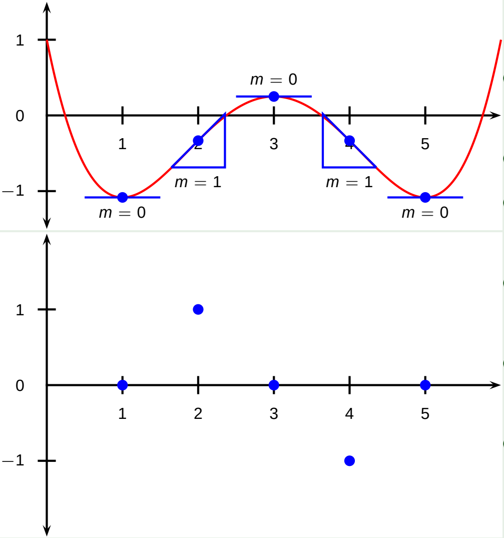

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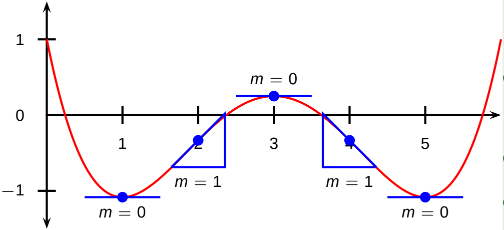

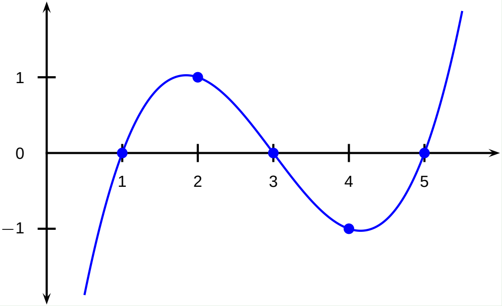

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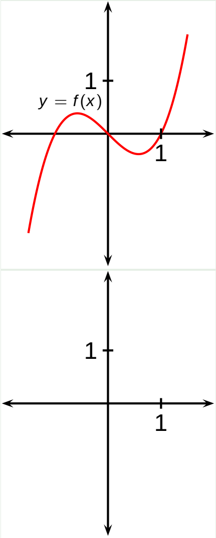
Example

If $f(x) = x^3 - x$, find the formula for $f'(x)$.



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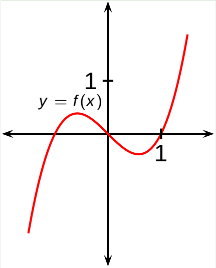

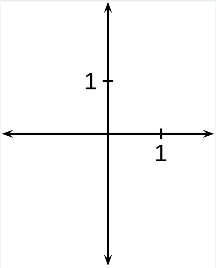

$$f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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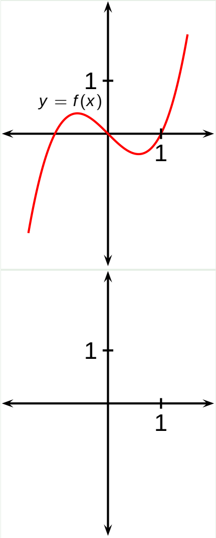

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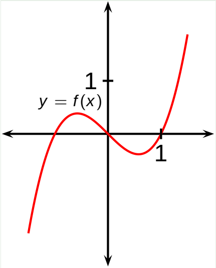

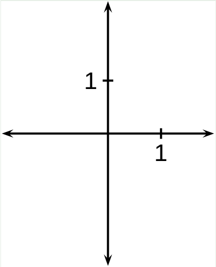

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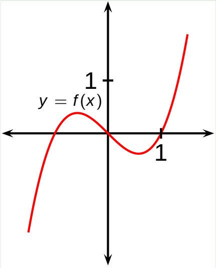

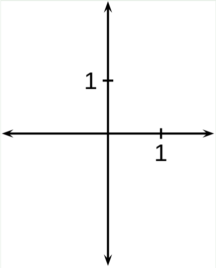

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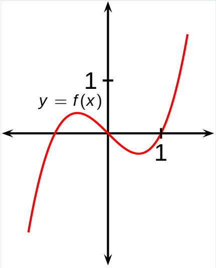

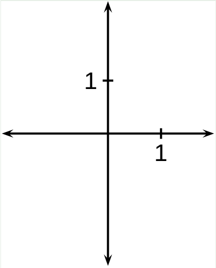

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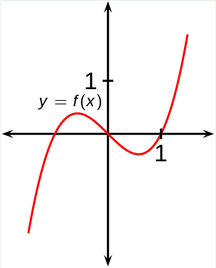

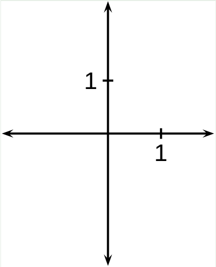

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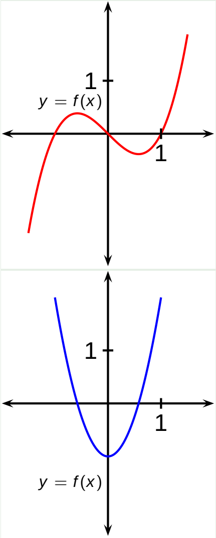

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Other Notations

If $y = f(x)$ is a function, there are many ways to write its derivative.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

- D and d/dx are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating the derivative.
- dy/dx is called Leibniz notation, and should not be seen as a ratio; it just means the same as $f'(x)$.
- If we want to indicate the value of the derivative dy/dx in Leibniz notation at a point a, we write

$$\left.\frac{dy}{dx}\right|_a \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_a$$