Math 140

Lecture 10

Greg Maloney

with modifications by T. Milev

University of Massachusetts Boston

March 5, 2013

Outline

1. (2.2)The Derivative as a Function
   - Differentiability
   - How Can a Function Fail to be Differentiable?
   - Higher Derivatives

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   - Differentiability
   - How Can a Function Fail to be Differentiable?
   - Higher Derivatives
2. How Derivatives Affect the Shape of a Graph
   - What Does f' Say About f?
   - What Does f'' Say About f?

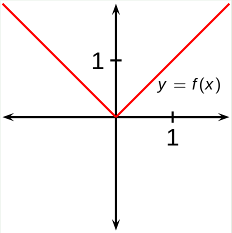
Outline

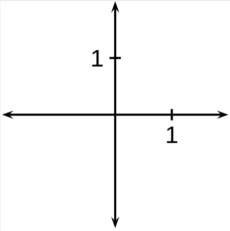
Definition (Differentiable)

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on the open interval (a, b) [or (a, ∞) or $(-\infty, b)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example

Where is the function $f(x) = |x|$ differentiable?





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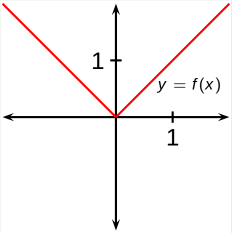

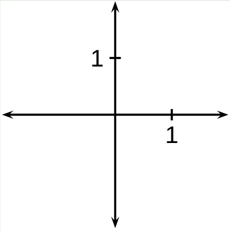

- Suppose $x > 0$.

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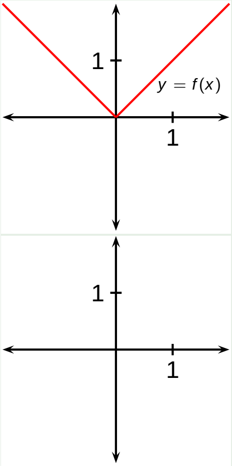

- Suppose $x > 0$.
- Then $|x| = x$.

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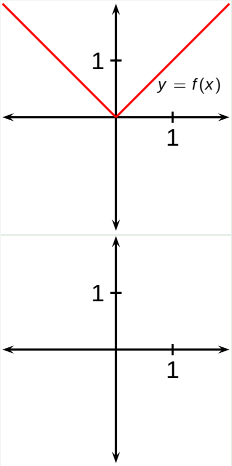

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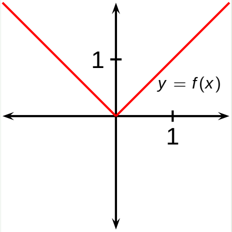

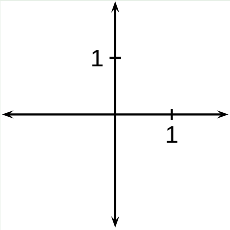

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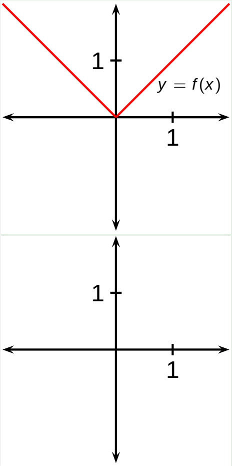

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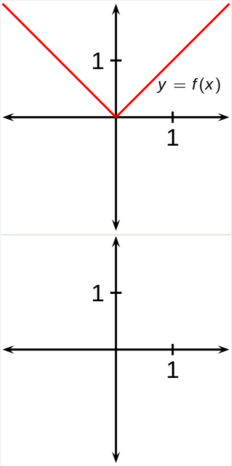

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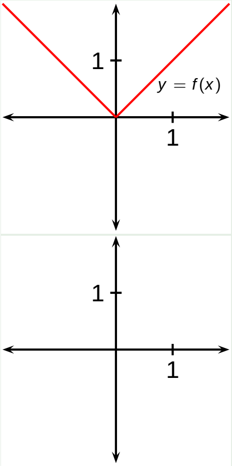

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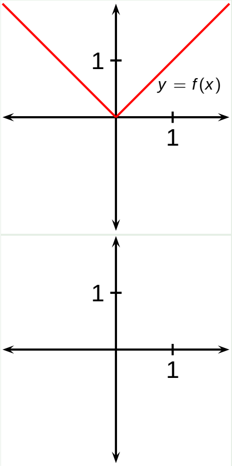

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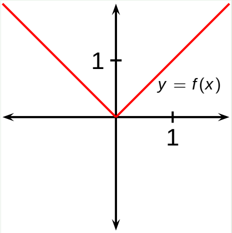

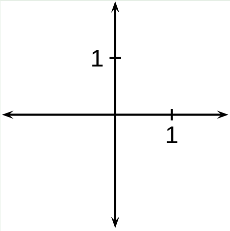

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Therefore f is differentiable for any $x > 0$.

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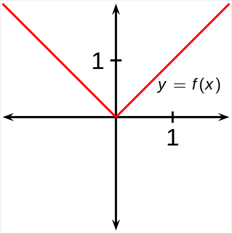

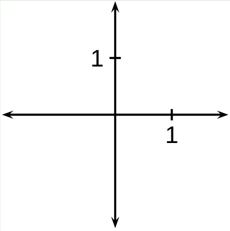

- Suppose $x < 0$.

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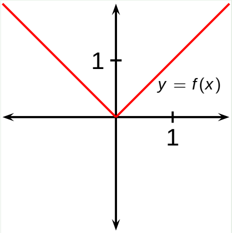

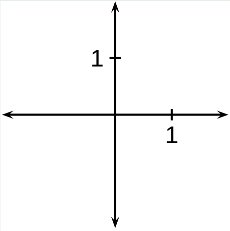

- Suppose $x < 0$.
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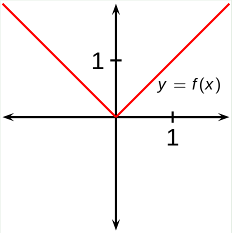

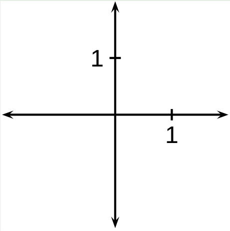

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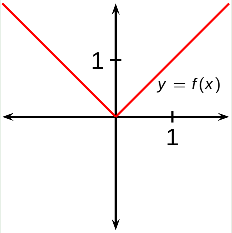

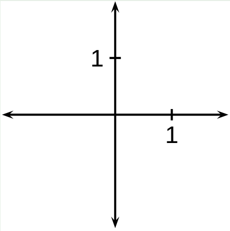

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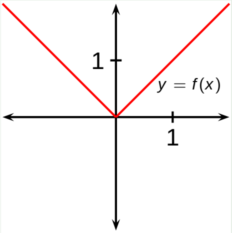

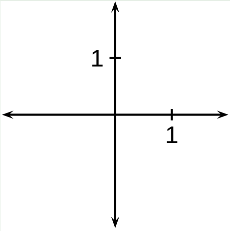

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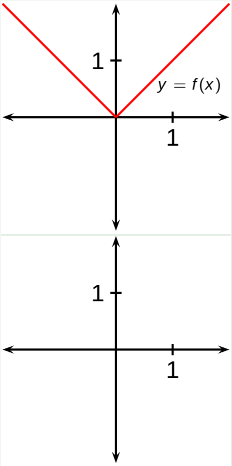

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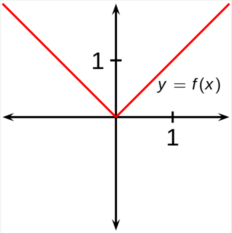

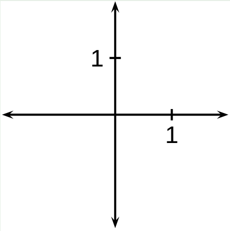

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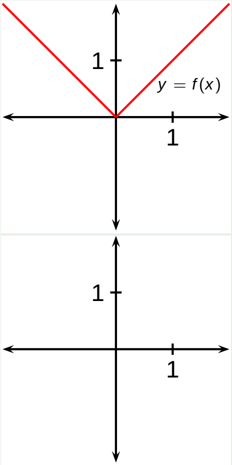

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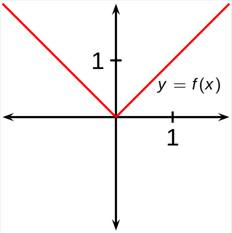

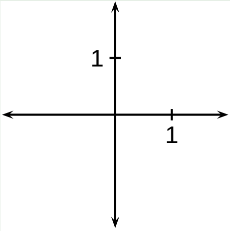

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Therefore f is differentiable for any $x < 0$.

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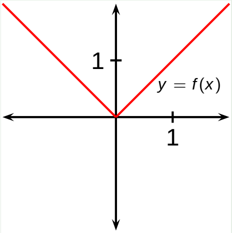

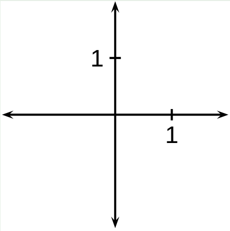

If $f'(0)$ exists, then

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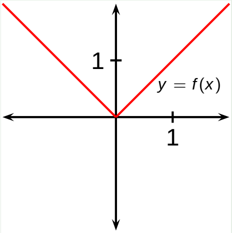

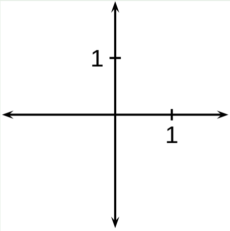

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Does this limit exist?

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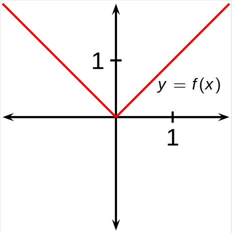

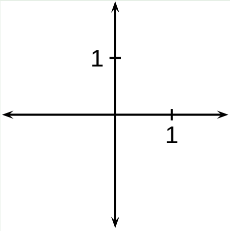

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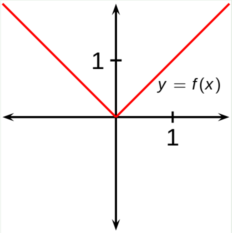

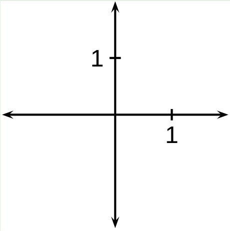

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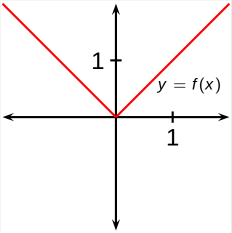

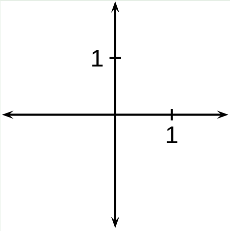

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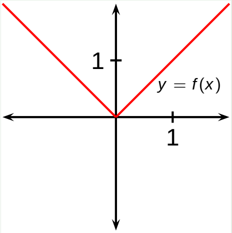

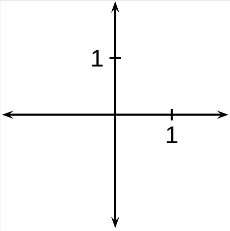

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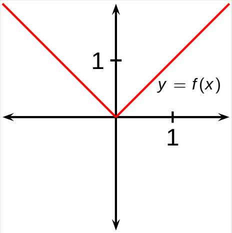

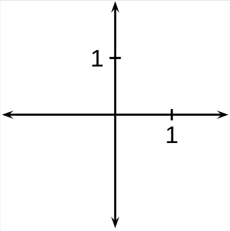

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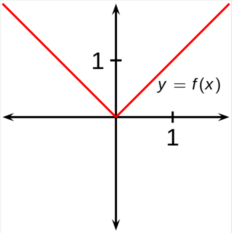

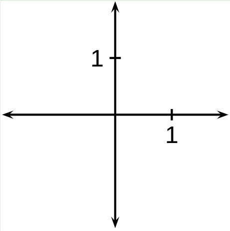

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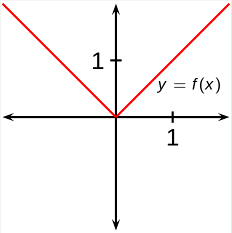

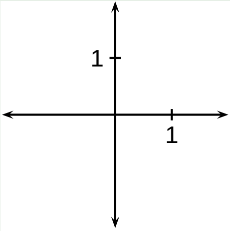

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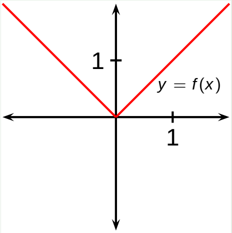
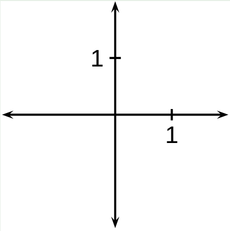
Where is the function $f(x) = |x|$ differentiable?





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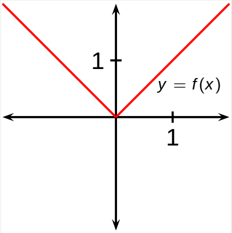

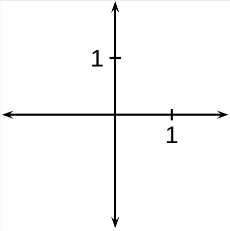

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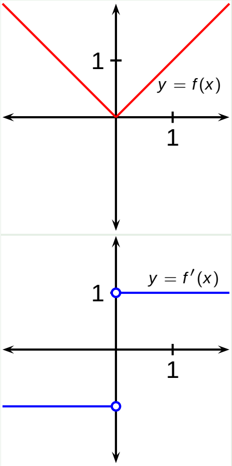

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$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

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If f is differentiable at a, then f is continuous at a.

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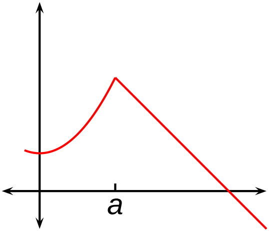

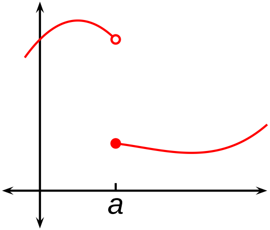

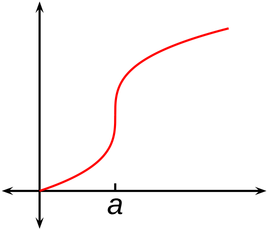

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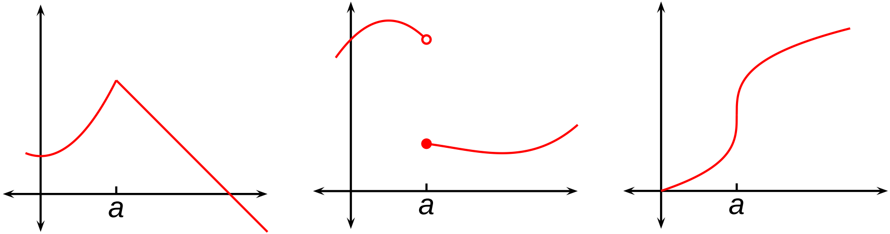

corner

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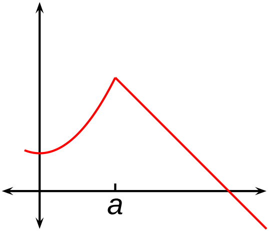

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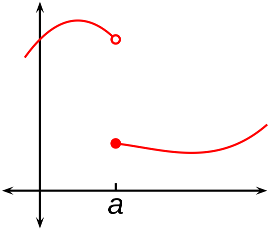

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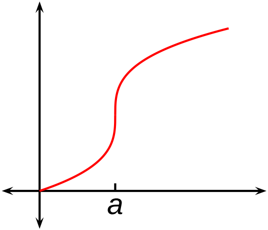

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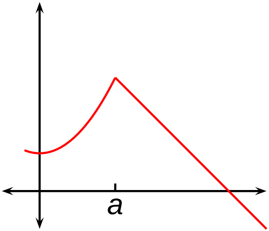

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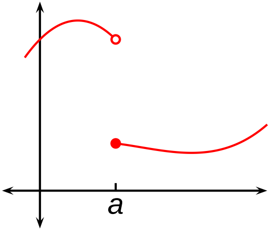

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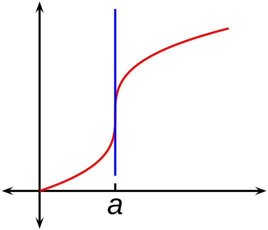

vertical tangent

Higher Derivatives

If f is a differentiable function, then f' is also a function, so f' might have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f.

In Leibniz notation, the second derivative of $y = f(x)$ is written

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

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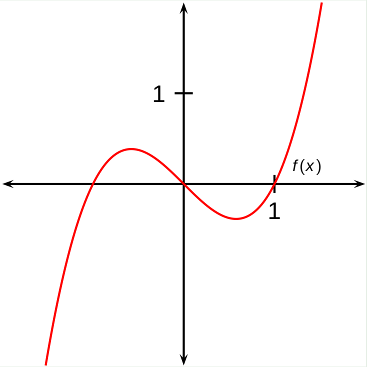
The fourth derivative is denoted by $f^{(4)}$, and for $n > 3$ the nth derivative is denoted by $f^{(n)}$.

Example

If $f(x) = x^3 - x$, find $f''(x)$.



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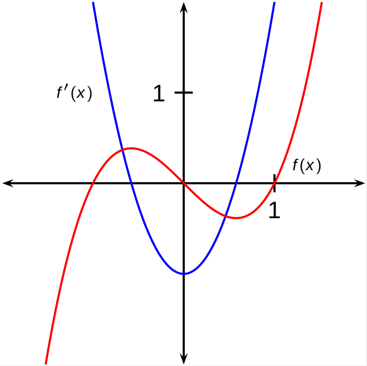

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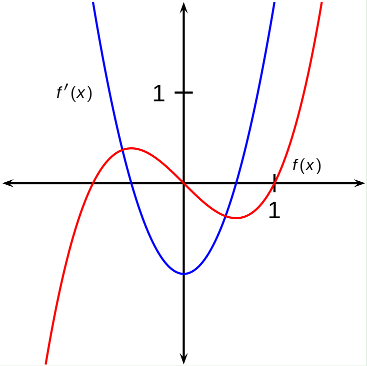

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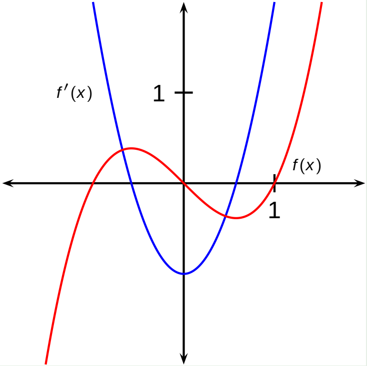

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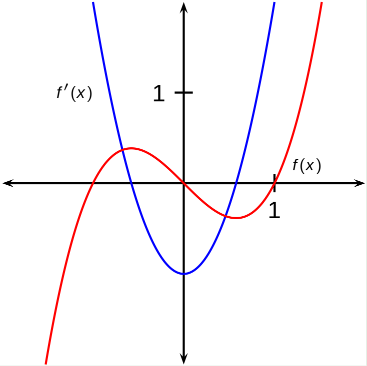

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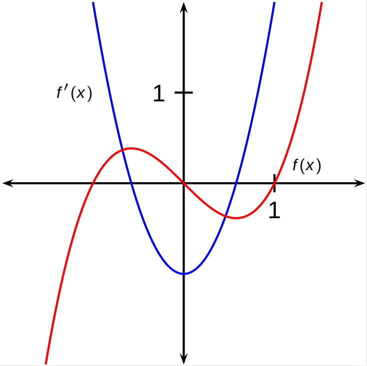

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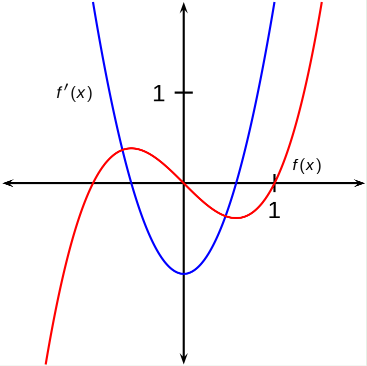

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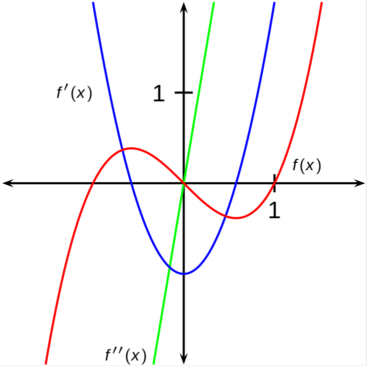

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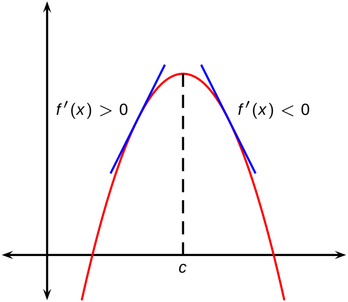
What Does f' Say About f?



- Consider the graph on the left.
- $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
- f is increasing to the left of c and decreasing to the right of c.

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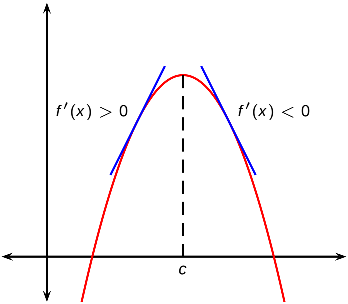

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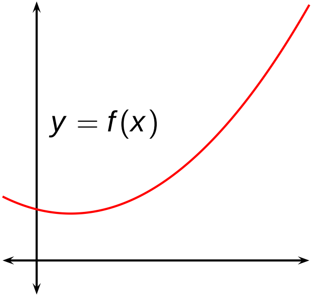
- This property holds more generally:

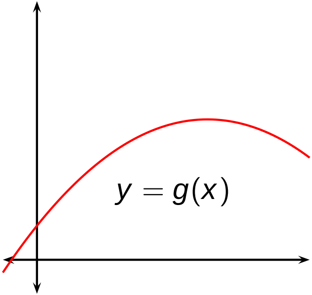
Increasing/Decreasing Test

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

What Does f'' Say About f?

f and g are both increasing functions on (a, b) with the same end points, but they look different because they bend in different directions.





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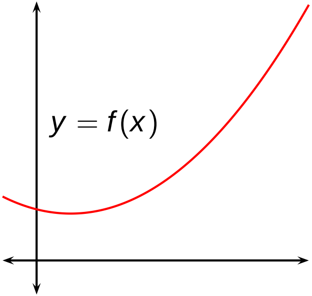

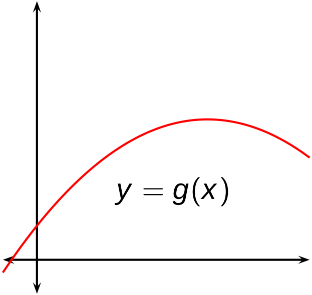

Definition (Concave Up/Concave Down)

If the graph of f lies above all of its tangents on an interval I, then it is called concave up on I. If it lies below all of its tangents on I, it is called concave down on I.

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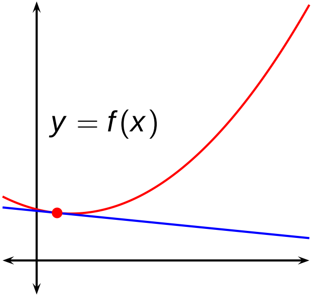

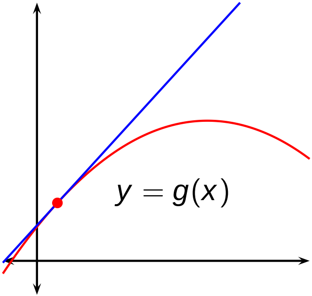

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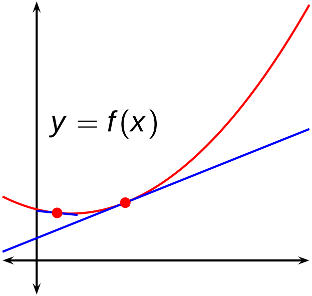

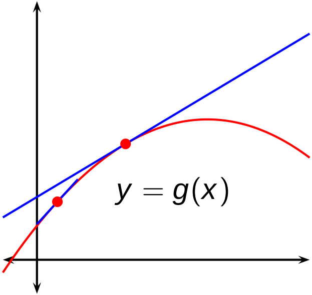

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If the graph of f lies above all of its tangents on an interval I, then it is called concave up on I. If it lies below all of its tangents on I, it is called concave down on I.

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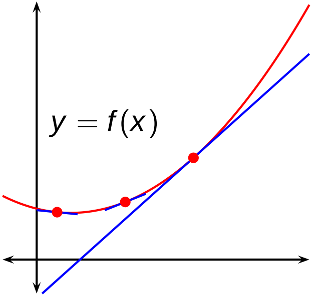

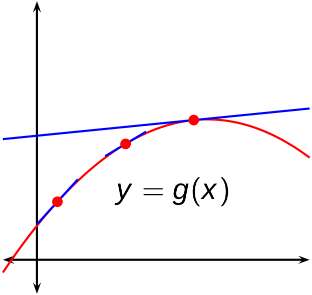

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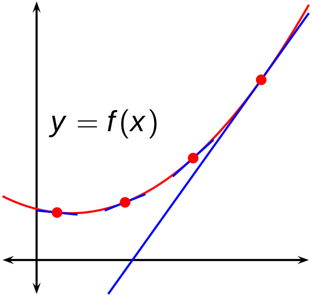

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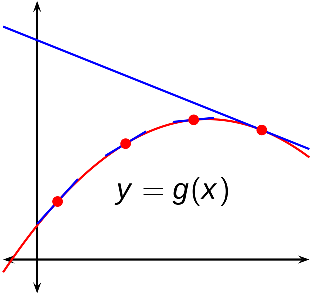

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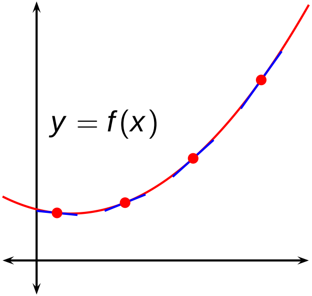

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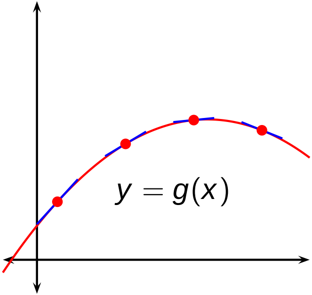

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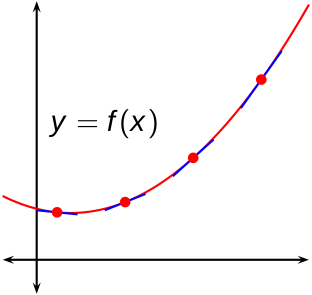

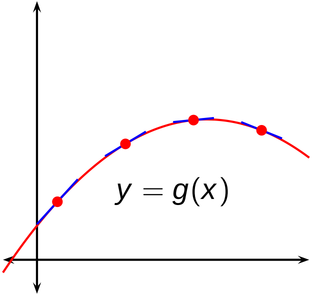

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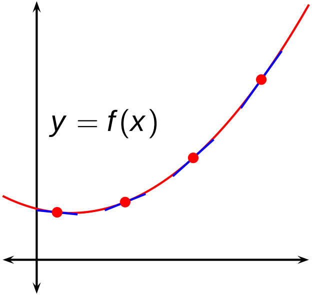

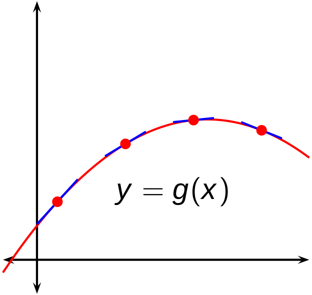

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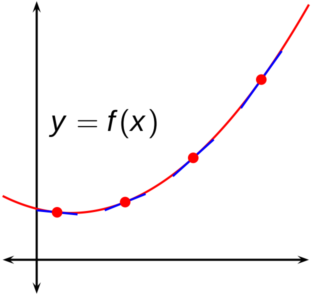
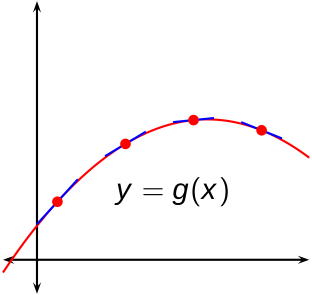
- This means f' is an increasing function.





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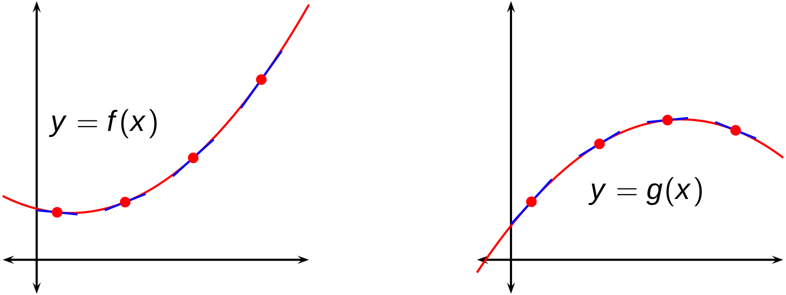

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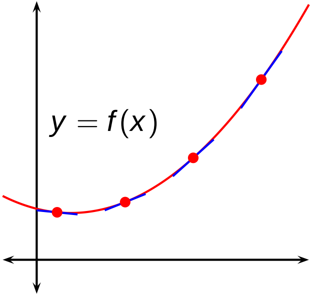

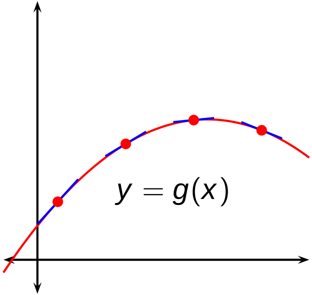

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Concavity Test

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Definition (Inflection Point)

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous at P and the curve changes from concave up to concave down or from concave down to concave up at P.

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Another way of saying this is that P is an inflection point if f'' changes signs at P.

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Let c be a constant and consider the constant function $f(x) = c$. Let us calculate the derivative of f:

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Theorem (Derivative of a Constant Function)

$$\frac{d}{dx}(c) = 0$$

Power Functions

Now consider functions of the form $f(x) = x^n$, where n is a positive integer. For $f(x) = x$, the graph is the line $y = x$, which has slope 1. So

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

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Theorem (The Power Rule)

If n is a positive integer, then $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof.

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Use this formula (which you can verify):

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□

Example (Power Rule)

If $f(x) = x^5$,

Then $f'(x) =$

If $y = x^{1000}$,

Then $y' =$

If $u = t^{22}$,

Then $\dfrac{du}{dt} =$

$$\frac{d}{dr}(r^3) =$$

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If $f(x) = x^5$,

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If $y = x^{1000}$,

Then $y' =$

If $u = t^{22}$,

Then $\frac{du}{dt} =$

$\frac{d}{dr}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,

Then $f'(x) = 5x^4$.

If $y = x^{1000}$,

Then $y' =$

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If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

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Then $y' = 1000x^{999}$.

If $u = t^{22}$,
Then $\frac{\mathrm{d}u}{\mathrm{d}t} = 22t^{21}$.

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) = 3r^2.$$

Derivative ball volume =surface area

The relationship between surface area and volume of a ball.

Dimension	Pts. at distance $\leq r$ from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
3						
2						
1						

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Dimension	Pts. at distance $\leq$ r from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
3						
2						
1						

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3	ball					
2						
1						

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3	ball					
2						
1						

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Dimension	Pts. at distance $\leq r$ from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
3	ball	ball volume				
2						
1						

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Dimension	Pts. at distance $\leq r$ from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
3	ball	ball volume				
2						
1						

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Dimension	Pts. at distance $\leq$ r from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
3	ball	ball volume	$\frac{4}{3}\pi r^3$			
2						
1						

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3	ball	ball volume	$\frac{4}{3}\pi r^3$			
2						
1						

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3	ball	ball volume	$\frac{4}{3}\pi r^3$	sphere surface area		
2						
1						

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2						
1						

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3	ball	ball volume	$\frac{4}{3}\pi r^3$	sphere surface area	$4\pi r^2$	
2						
1						

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2						
1						

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2						
1						

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2						
1						

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2	disk, circle					
1						

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1						

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2	disk, circle	circle area				
1						

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1						

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1						

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1						

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1						

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2	disk, circle	circle area	πr^2	circle circumference	$2\pi r$	
1						

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1	interval					

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1	interval	length				

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2	disk, circle	circle area	πr^2	circle circumference	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	interval	length	$2r$			

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1	interval	length	$2r$	the two endpoints		

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2	disk, circle	circle area	πr^2	circle circumference	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	interval	length	$2r$	the two endpoints	2	

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Dimension	Pts. at distance $\leq r$ from origin	Inside-volume name	Inside-volume f-la	Boundary name	Boundary area f-la	Derivative inside-volume
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1	interval	length	$2r$	the two endpoints	2	$\frac{d}{dr}(2r) =$

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2	disk, circle	circle area	πr^2	circle circumference	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	interval	length	$2r$	the two endpoints	2	$\frac{d}{dr}(2r) = 2$