Exam I
Calculus I, Math 140
March 7, 2013

Name:

Problem 1 *(7.5 + 7.5 pts) Find an expression for the function* $(f \circ g)(x)$ *and* $(g \circ f)(x)$*, where*

$$f(x) = \frac{3x-1}{x-2}, \quad g(y) = \frac{y-2}{2y-4}$$

Simplify your answer to a single fraction.

Solution. Note that for $y \neq 2$, we have that $g(y) = \frac{y-2}{2y-4} = \frac{\cancel{(y-2)}}{2\cancel{(y-2)}} = \frac{1}{2}$. Therefore

$$f(g(x)) = f(\frac{1}{2}) = \frac{3 \times \frac{1}{2} - 1}{\frac{1}{2} - 2} = \frac{\frac{1}{2}}{-\frac{3}{2}} = -\frac{1}{3} \quad .$$

Furthermore, since $g(y) = \frac{1}{2}$, we need not compute anything to get a formula for $g(f(x))$ - indeed, the value of g does not depend on its argument:

$$g(f(x)) = \frac{1}{2} \quad .$$

Final answer: $(f \circ g)(x) = -\frac{1}{3}$, $(g \circ f)(x) = \frac{1}{2}$.

Problem 2 *(15 pts) Find all solutions in the interval* $[0, 2\pi]$ *of the equation.* $\sqrt{3} \sin x = \sin 2x$

Solution. We have the formula $\sin 2x = 2 \sin x \cos x$. Therefore

$$\begin{aligned} \sqrt{3} sinx &= 2 \sin x \cos x \\ \sqrt{3} \sin x - 2 \sin x \cos x &= 0 \\ \sin x(\sqrt{3} - 2\cos x) &= 0 \quad . \end{aligned}$$

Thus either $\sin x = 0$ or $\sqrt{3} - 2\cos x = 0$. As studied in the lecture on trigonometry, for $x \in [0, 2\pi]$, we have that $\sin x = 0$ if $x = 0, \pi$ or 2π. On the other hand $\sqrt{3} - 2\cos x = 0$ is equivalent to $\cos x = \frac{\sqrt{3}}{2}$, which, together with $x \in [0, 2\pi]$, implies $x = \frac{\pi}{6}$ or $x = \frac{11\pi}{6}$.

Final answer: $x \in \{0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi\}$.

Problem 3 *(15 pts) Evaluate the limit if it exists.*

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^3 - 4x}$$

Solution.

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^3 - 4x} = \lim_{x \to 2} \frac{(3x+2)\cancel{(x-2)}}{x\cancel{(x-2)}(x+2)} = \lim_{x \to 2} \frac{3x+2}{x(x+2)} = \frac{3 \times 2 + 2}{2(2+2)} = \frac{8}{8} = 1.$$

Problem 4 *(10 pts) Evaluate the limit.*

$$\lim_{x \to 2^+} \frac{\sqrt{x^3 - 4x}}{3x^2 - 4x - 4} \quad .$$

Solution.

$$\lim_{x \to 2^+} \frac{\sqrt{x^3 - 4x}}{3x^2 - 4x - 4} = \lim_{x \to 2^+} \frac{(x(x-2)(x+2))^{\frac{1}{2}}}{(3x+2)(x-2)} = \lim_{x \to 2^+} \frac{x^{\frac{1}{2}}(x+2)^{\frac{1}{2}}}{(3x+2)(x-2)^{\frac{1}{2}}} \quad .$$

As the denominator of the above fraction tends to 0 without changing sign as x tends to 2, and the numerator is non-zero, the limit must be equal to either ∞ or $-\infty$. On the other hand $x \to 2^+$ implies that $x > 2$ and therefore multiplicands in the above limit are positive. Therefore the limit equals $+\infty$.

Final answer: $\lim_{x \to 2^+} \frac{\sqrt{x^3 - 4x}}{3x^2 - 4x - 4} = \infty$.

Problem 5 *(10 pts) Use the intermediate value theorem to show that the equation has a solution in the interval* $(-2, 0)$.

$$e^{2x} + x + 1 = \sin(-x) \quad . \tag{1}$$

Solution. Let $f(x) = e^{2x} + x + 1 - \sin(-x) = e^{2x} + x + 1 + \sin(+x)$. A number x is a solution to the equation (1) if and only if $f(x) = 0$. On the other hand $f(-2) = e^{-4} - 2 + 1 + \sin(-2) = 1 - e^{-4} - \sin 2$. We know that $e^{-4} = \frac{1}{e^4} < 1$ and so $1 - e^4 < 0$. Furthermore as $0 < 2 < \pi$ we have that $\sin 2 > 0$ and $-\sin 2 < 0$ (recall that $\sin 2$ is measured in radians). Therefore $1 - e^{-4} - \sin 2 < 0$. On the other hand, $f(0) = e^0 + 0 + 1 + \sin 0 = 2 > 0$. We have that $f(x)$ is continuous: we studied in class that $1, x, \sin x$ and e^x are all continuous functions, and we also studied that sum of continuous functions is continuous. Furthermore $f(-2) < 0 < f(0)$. Therefore by the Intermediate Value Theorem we have that there exists a number $c \in (-2, 0)$ for which $f(c) = 0$, which proves that (1) has a solution in the desired interval.

Problem 6 *(10pts) Solve the equation.*

$$e^{-4x} + 5e^{-2x} - 6 = 0 \quad .$$

Set $z = e^{-2x}$. The equation becomes

$$\begin{array}{rcl} z^2 + 5z - 6 &=& 0 \\ (z+6)(z-1) &=& 0 \quad . \end{array}$$

Therefore $e^{-2x} = z = 1$ or $e^{-2x} = z = -6$. The latter case, $e^{-2x} = -6$, is not possible for real x. Therefore $e^{-2x} = 1$, and $-2x = \ln 1 = 0$, so $x = 0$.

Final answer: $x = 0$.

Problem 7 *(15 pts) Plot roughly the function*

$$f(x) = x^2 + 2x + 2$$

with domain $x > -1$. Compute the inverse function $f^{-1}(y)$. Plot roughly $f^{-1}(x)$. Explain what is the relationship between the graphs of $f^{-1}(x)$ and $f(x)$.

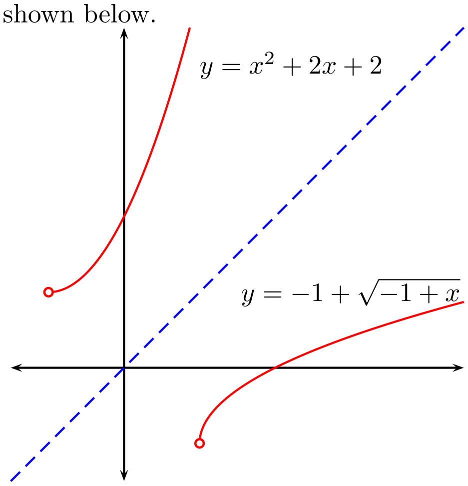
To compute $f^{-1}(y)$, we solve the equation $y = f(x)$ for y:

$$\begin{array}{rcl} x^2 + 2x + 2 &=& y \\ x^2 + 2x + 2 - y &=& 0 \\ (x+1)^2 &=& y - 1 \\ x + 1 &=& \pm\sqrt{y-1} \\ x &=& -1 \pm \sqrt{y-1} \quad . \end{array}$$

On the other hand we are given that $x > -1$, and therefore, as $\sqrt{y-1} > 0$ (square roots are positive by definition), we have that $x = f^{-1}(y) = -1 + \sqrt{y-1}$, $y > 1$. Therefore, after relabelling the dummy argument variable of f^{-1} from y to x, we get

$$f^{-1}(x) = -1 + \sqrt{x-1}, \qquad x > 1 \quad .$$

The function $f(x)$, $x > -1$ is easy to plot - that is half of a parabola with a vertex at $x = -1$, $y = f(-1) = 1$. On the other hand, the graph of $f^{-1}(x)$ is the reflection of $f(x)$ across the line $y = x$. This is easy to plot roughly by hand, as expected of you on the test. Using the explicit formulas for $f(x)$ and $f^{-1}(x)$ above, it is very easy to plot the functions by computer, as shown below.



Problem 8 *(10 pts) Compute the limit.*

$$\lim_{x \to \infty} \sqrt{x^2 + 3x} - \sqrt{x^2 - \frac{x}{3}} \quad .$$

Solution.

$$\begin{aligned}
\lim_{x\to\infty}\sqrt{x^2+3x}-\sqrt{x^2-\tfrac{x}{3}} &= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+3x}-\sqrt{x^2-\frac{x}{3}}\right)\left(\sqrt{x^2+3x}+\sqrt{x^2-\frac{x}{3}}\right)}{\sqrt{x^2+3x}+\sqrt{x^2-\frac{x}{3}}} = \lim_{x\to\infty}\frac{\sqrt{x^2+3x}^2-\sqrt{x^2-\frac{x}{3}}^2}{\sqrt{x^2+3x}+\sqrt{x^2-\frac{x}{3}}}\\
&= \lim_{x\to\infty}\frac{\cancel{x^2}+3x\cancel{-x^2}+\frac{x}{3}}{\sqrt{x^2+3x}+\sqrt{x^2-\frac{x}{3}}} = \lim_{x\to\infty}\frac{\frac{10}{3}\cancel{x}\cancel{\frac{1}{x}}}{\left(\sqrt{x^2+3x}+\sqrt{x^2-\frac{x}{3}}\right)\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{10}{3}}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{1}{3x}}}\\
&= \frac{\frac{10}{3}}{\sqrt{1+\lim\limits_{x\to\infty}\frac{3}{x}}+\sqrt{1-\lim\limits_{x\to\infty}\frac{1}{3x}}} = \frac{\frac{10}{3}}{\sqrt{1+0}+\sqrt{1-0}} = \frac{10}{6} = \frac{5}{3} \quad .
\end{aligned}$$

Problem 9 *(15 pts) Compute the limit.*

$$\lim_{x\to\infty}\sqrt{3^{2x}+3^{x+1}}-\sqrt{3^{2x}-3^{x-1}} \quad .$$

Solution. The function mapping x to 3^x is one to one and increasing. Furthermore, as $x\to\infty$, we have hat $3^x\to\infty$. Therefore we have the right to change variables $u=3^x$. Therefore we have that

$$\lim_{\substack{x\to\infty\\ u=3^x\\ u\to\infty}}\sqrt{3^{2x}+3^{x+1}}-\sqrt{3^{2x}-3^{x-1}} = \lim_{3^x=u\to\infty}\sqrt{(3^x)^2+3\times 3^x}-\sqrt{(3^x)^2-\frac{3^x}{3}} = \lim_{u\to\infty}\sqrt{u^2+3u}-\sqrt{u^2-\frac{u}{3}} = \frac{5}{3} \quad ,$$

where we have noticed that last limit is identical to that given in Problem 8, and we have already computed it in the solution to the previous problem.