Math 140

Lecture 12

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Outline

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Now we need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

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Therefore $f'(x)g'(x) \neq (fg)'(x)$.

The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof.

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The proof of the Quotient Rule uses a trick similar to the one in the proof of the Product Rule.

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If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

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$$= \frac{\left(-x^6 + 2\right)\left(\qquad\qquad\right) - \left(x^5 + 2x\right)\left(\qquad\qquad\right)}{\left(-x^6 + 2\right)^2}$$

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$$= \frac{\left(-x^6 + 2\right) \left(5x^4 + 2\right) - \left(x^5 + 2x\right) \left(\qquad \right)}{\left(-x^6 + 2\right)^2}$$

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$$= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2}$$

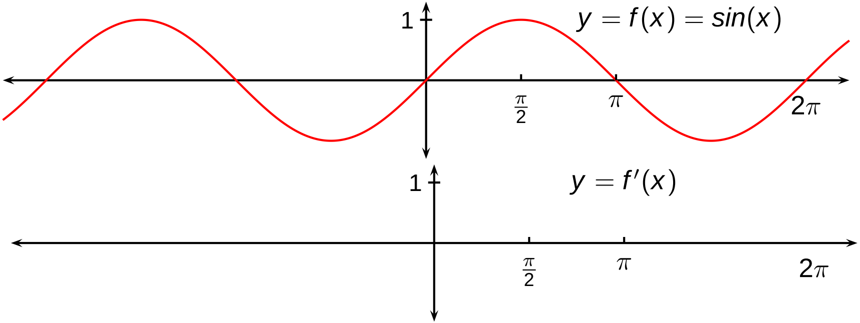
Example (Quotient Rule, rational function)

Differentiate $y = \dfrac{x^5+2x}{-x^6+2}$.

Quotient Rule:

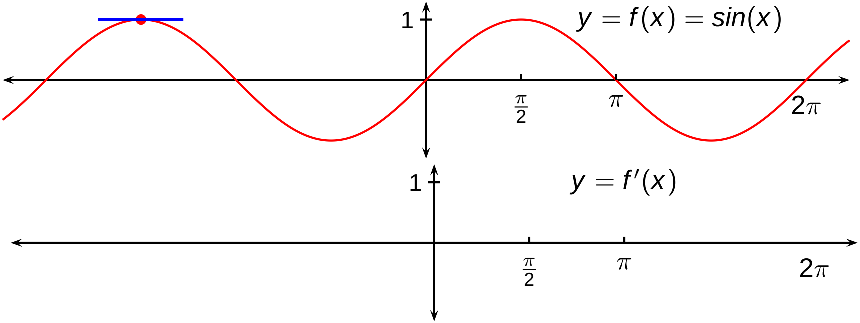
$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(-x^6+2\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5+2x\right)-\left(x^5+2x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^6+2\right)}{\left(-x^6+2\right)^2}\\
&= \frac{\left(-x^6+2\right)\left(5x^4+2\right)-\left(x^5+2x\right)\left(-6x^5\right)}{\left(-x^6+2\right)^2}\\
&= \frac{\left(-5x^{10}-2x^6+10x^4+4\right)-\left(-6x^{10}-12x^6\right)}{(-x^6+2)^2}\\
&= \frac{x^{10}+10x^6+10x^4+4}{(-x^6+2)^2}.
\end{aligned}
$$

Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$?

Derivatives of Trigonometric Functions



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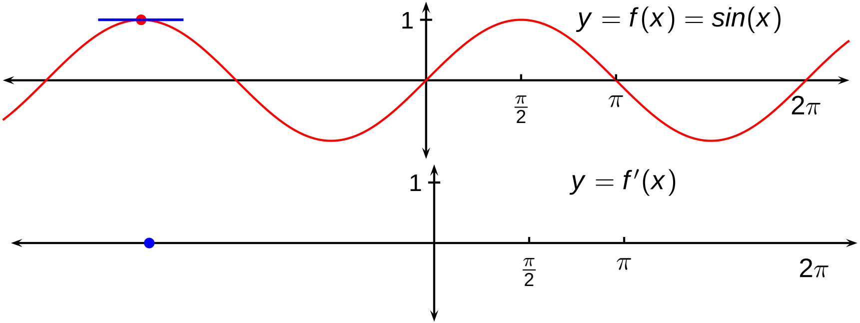
Derivatives of Trigonometric Functions



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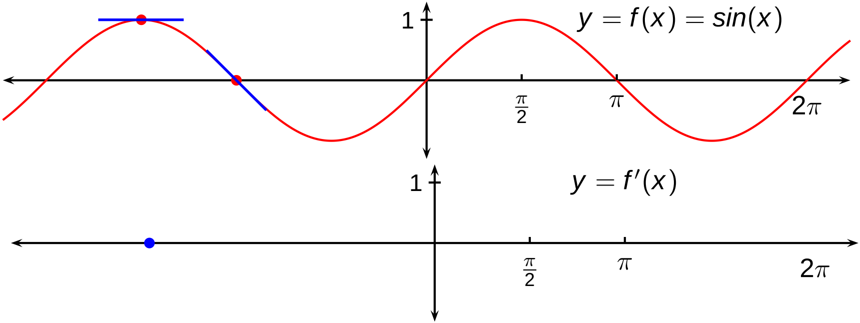
Derivatives of Trigonometric Functions



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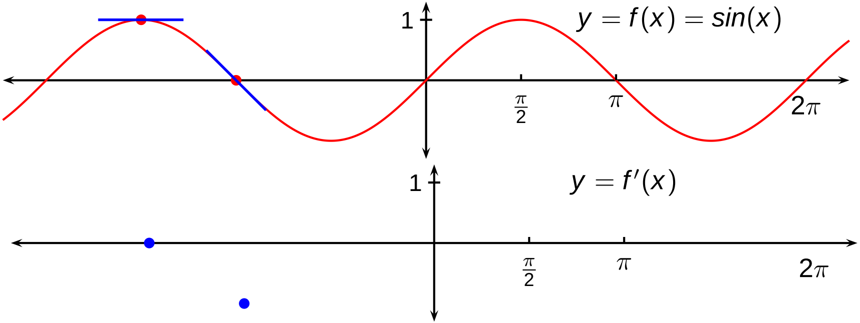
Derivatives of Trigonometric Functions



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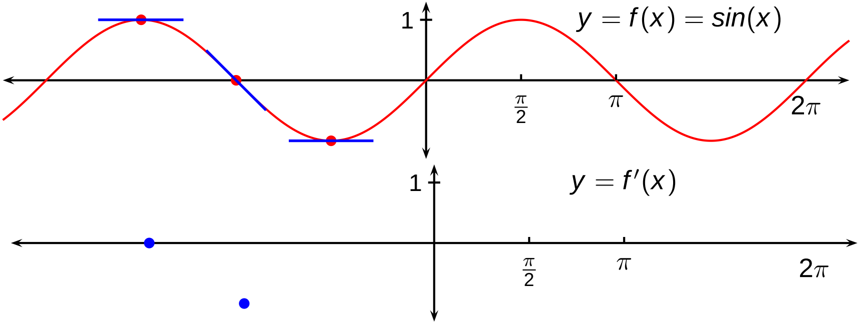
Derivatives of Trigonometric Functions



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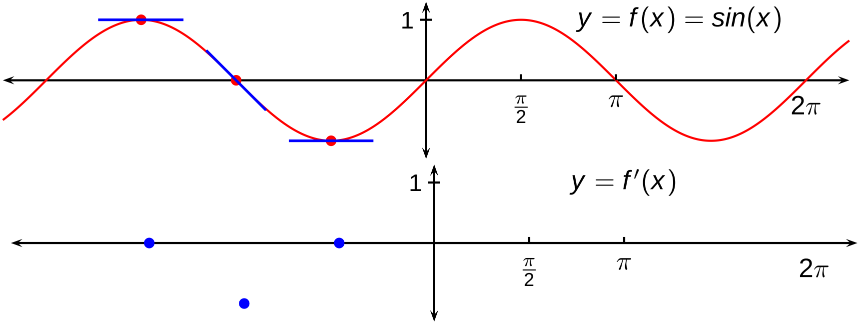
Derivatives of Trigonometric Functions



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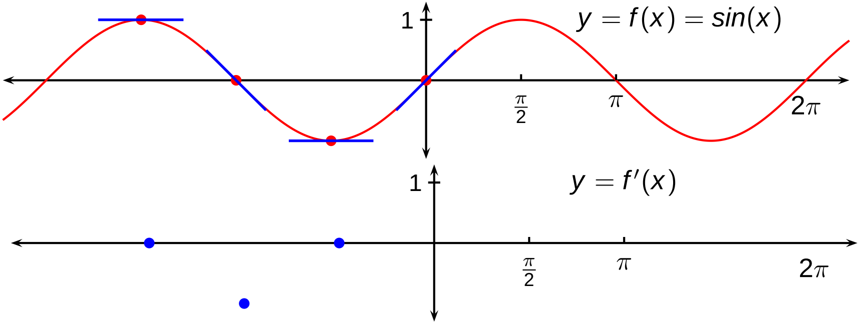
Derivatives of Trigonometric Functions



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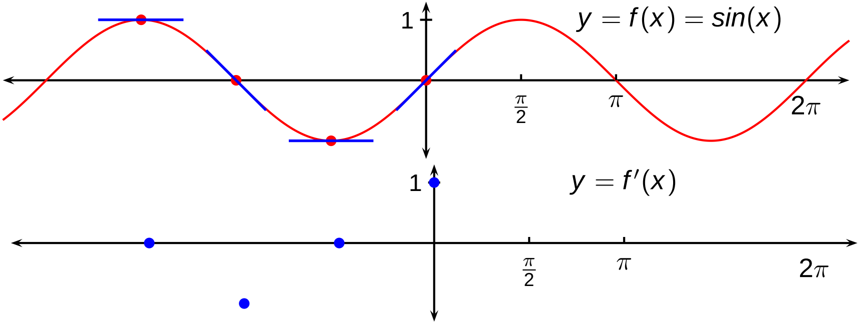
Derivatives of Trigonometric Functions



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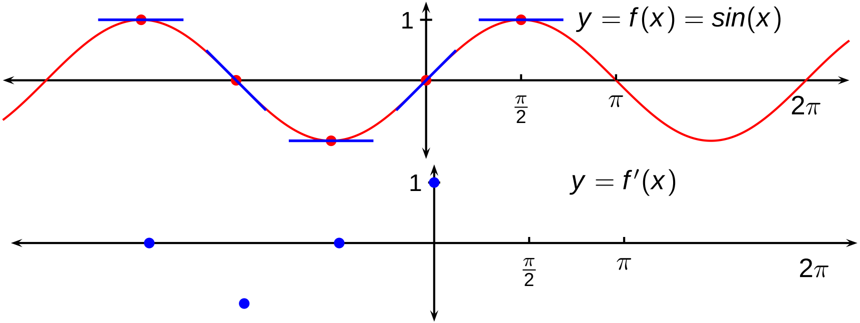
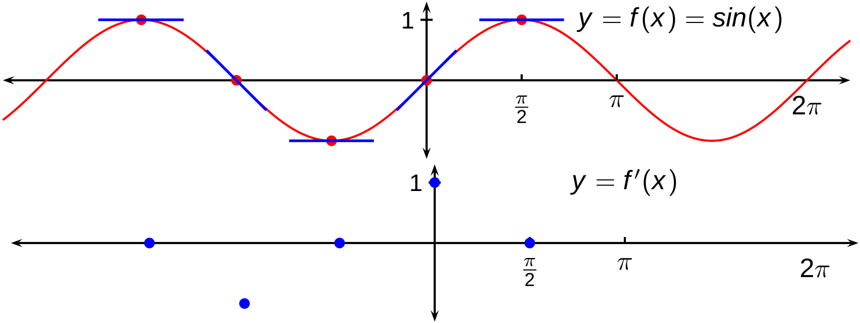
Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$?

Derivatives of Trigonometric Functions



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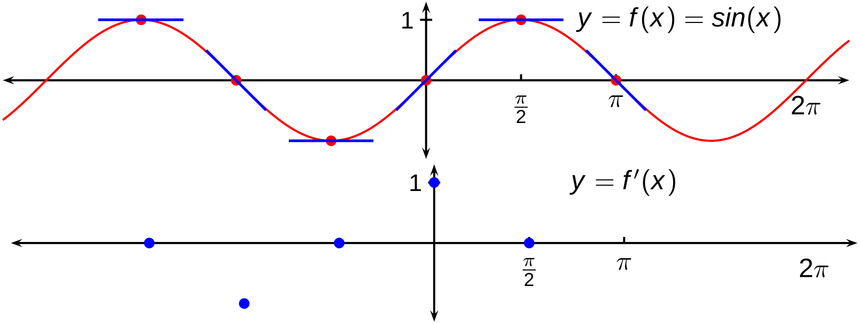
Derivatives of Trigonometric Functions



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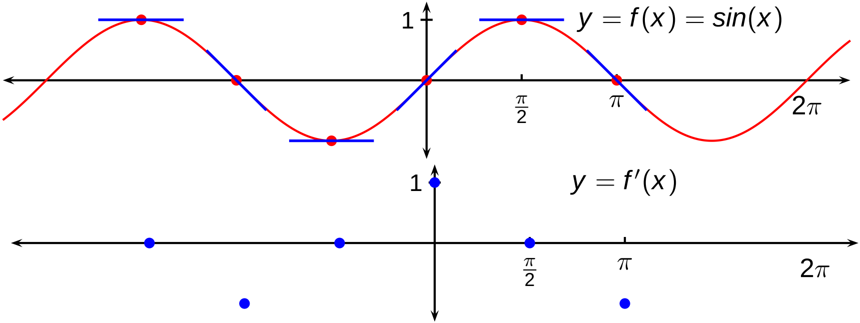
Derivatives of Trigonometric Functions



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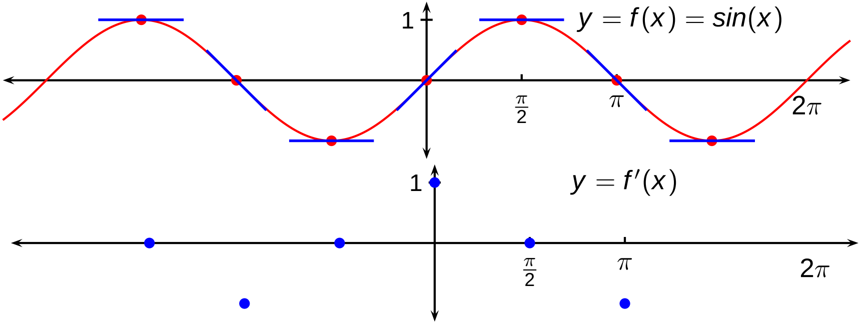
Derivatives of Trigonometric Functions



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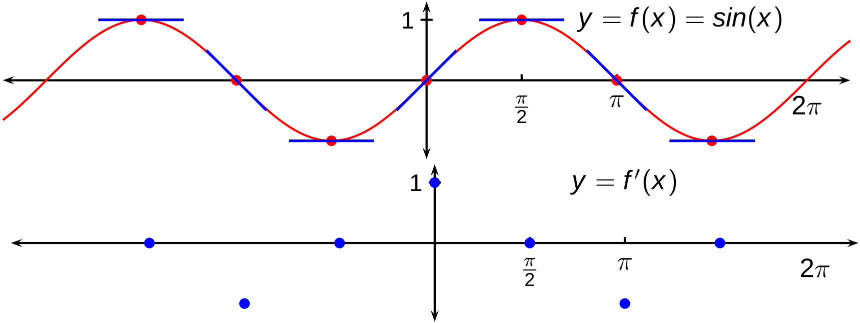
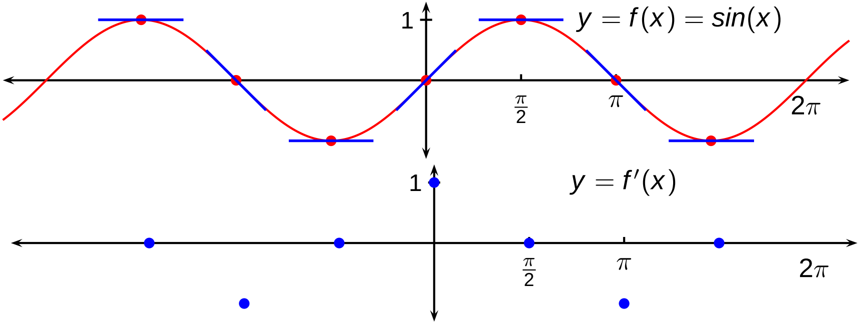
Derivatives of Trigonometric Functions



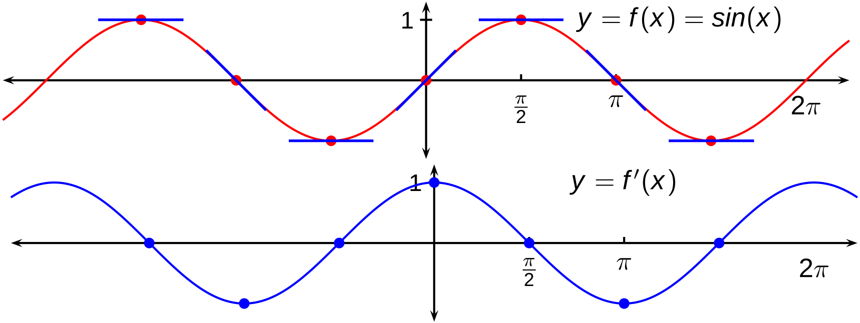
What is the derivative of $f(x) = \sin x$?

Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

Let $\quad f(x) = \sin x.$

Then $\quad f'(x) =$

$$\text{Let} \quad f(x) = \sin x.$$

$$\text{Then} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = \sin x.$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

Let $f(x) = \sin x.$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$\text{Let} \quad f(x) = \sin x.$$

$$\begin{aligned}
\text{Then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]
\end{aligned}$$

$$\text{Let} \quad f(x) = \sin x.$$

$$\begin{aligned}
\text{Then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]
\end{aligned}$$

$$\text{Let}\quad f(x) = \sin x.$$

$$\begin{aligned}
\text{Then}\quad f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\
&= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right]\\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]\\
&= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}
\end{aligned}$$

Let $f(x) = \sin x.$

Then
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
&= {\color{red}\lim_{h\to 0} \sin x} \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \quad \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \quad \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}$$

Let $f(x) = \sin x$.

Then
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$
$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$
$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$
$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \qquad \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Let $f(x) = \sin x.$

Then
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \qquad \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}$$

$$\text{Let} \quad f(x) = \sin x.$$

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\text{Then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
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&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}$$

$$
\begin{aligned}
\text{Let} \quad f(x) &= \sin x. \\
\text{Then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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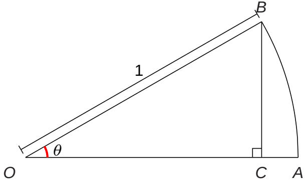
&= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}
$$

We need to do more work to find the other two limits.

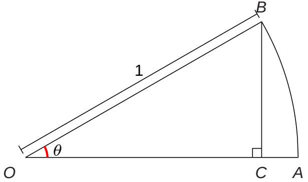
Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin \theta$ using ratios of side lengths of a triangle.



Claim: $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



$\sin\theta =$

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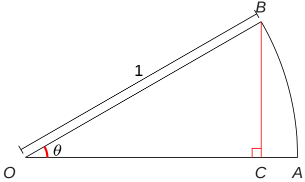
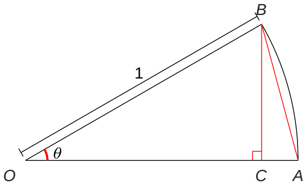


$$\sin \theta = |BC|$$

Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

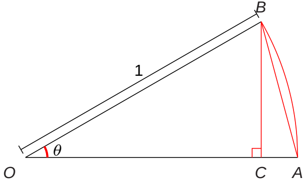
First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = |BC| < |AB|$$

Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

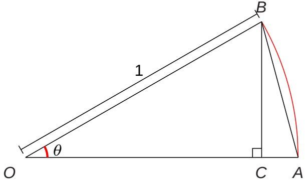
First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = |BC| < |AB| < \text{arc}AB$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

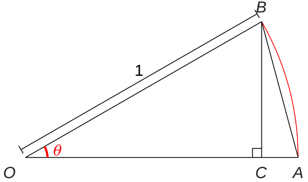
First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = |BC| < |AB| < \text{arc}AB =$$

Claim: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

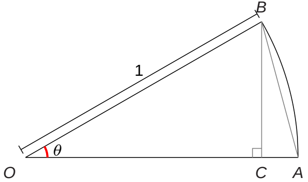
First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

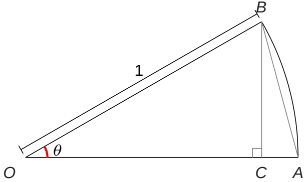
First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin \theta}{\theta} < 1$.

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.

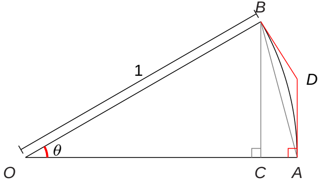
$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \text{arc}AB$$

Claim: $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin \theta$ using ratios of side lengths of a triangle.



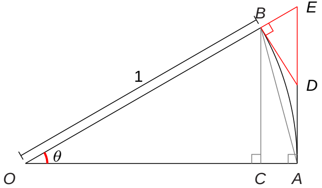
$$\sin \theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin \theta}{\theta} < 1$.

$$\theta = \text{arc}AB < |AD| + |DB|$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



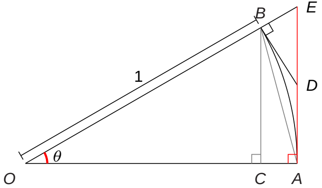
$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \text{arc}AB < |AD| + |DB| < |AD| + |DE|$$

Claim: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

First suppose $0<\theta<\frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



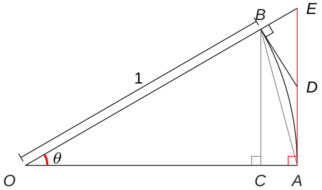
$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin\theta}{\theta}<1$.

$$\begin{aligned}\theta = \text{arc}AB &< |AD|+|DB| < |AD|+|DE|\\ &= |AE|\end{aligned}$$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



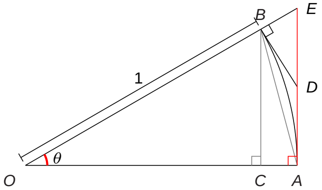
$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \text{arc}AB < |AD| + |DB| < |AD| + |DE|$$
$$= |AE| = |OA|\tan\theta$$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



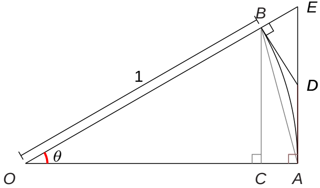
$$\sin\theta = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\frac{\sin\theta}{\theta} < 1$.

$$\begin{aligned}\theta = \text{arc}AB &< |AD| + |DB| < |AD| + |DE| \\ &= |AE| = |OA|\tan\theta = \tan\theta\end{aligned}$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

First suppose $0 < \theta < \frac{\pi}{2}$. Then we can write $\sin\theta$ using ratios of side lengths of a triangle.



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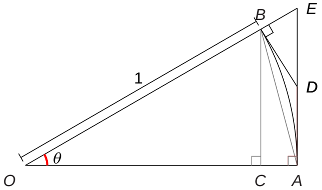

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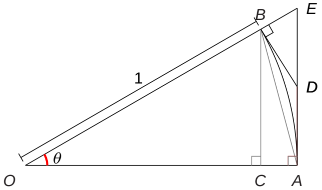

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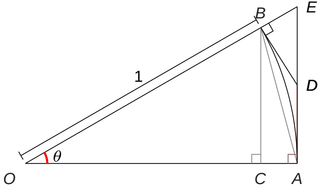

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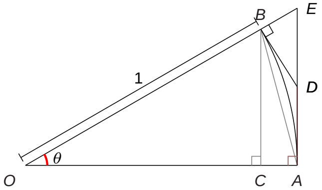

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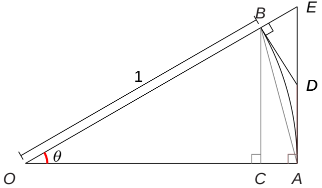

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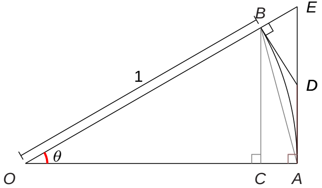

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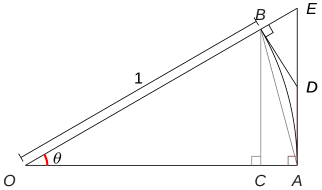

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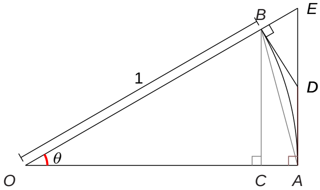

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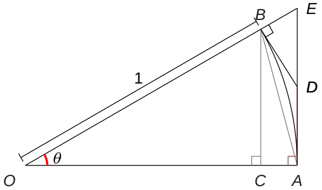

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Let $\quad f(x) = \sin x.$

Then $\quad f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$

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Theorem (The Derivative of $\sin x$)

$$\frac{d}{dx}\sin x = \cos x$$

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Differentiate $f(x) = x \sin x.$

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&= (x) \left(\cos x \right) + (\sin x) \left(1 \right)
\end{aligned}
$$

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&= (x) \ (\cos x) \ + (\sin x) \ (1) \\
&= x \cos x + \sin x.
\end{aligned}$$

Example (Quotient Rule, Natural Exponential Function and Sine)

Differentiate $y = \dfrac{e^x}{2 + \sin x}$.

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Differentiate $y = \dfrac{e^x}{2 + \sin x}$.

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Example (Trigonometric limit)

Find $\displaystyle \lim_{x\to 0} \frac{2x}{\sin 9x}$

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The same techniques we used to find the derivative of $\sin x$ can also be used to find the derivative of $\cos x$.

Theorem (The Derivative of $\cos x$)

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Example (Product Rule, with Cosine)

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Theorem (The Derivative of Tangent)

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

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