Math 140

Lecture 13

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Outline

1. (2.4) Derivatives of Trigonometric Functions

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2. (2.5) The Chain Rule

Derivatives of Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

Example (Quotient Rule,)

Differentiate $y = \dfrac{\sec x}{1 + \tan x}$.

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&= \frac{(1+\tan x)\left(\sec x \tan x\right) - (\sec x)\left(\sec^2 x\right)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.
\end{aligned}$$

Example (Using the Product Rule twice)

Differentiate:

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Find the 27th derivative of $f(x) = \cos x$.

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f'(x) &= \\
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- $y = f(u) = \qquad .$
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- We know the derivatives of f and g:
- $f'(u) =$.
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- We know the derivatives of f and g:
- $f'(u) = \frac{1}{2}u^{-1/2}$.
- $g'(x) = \quad$.

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- We know the derivatives of f and g:
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- It would be nice if we could find the derivative of F in terms of the derivatives of y and u.

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- It turns out that the derivative of the composition $f \circ g$ is the product of the derivative of f and the derivative of g.

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- $u = g(x) = x^2 + 1$.
- Then $y = F(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$.
- We know the derivatives of f and g:
- $f'(u) = \frac{1}{2}u^{-1/2}$.
- $g'(x) = 2x$.
- It would be nice if we could find the derivative of F in terms of the derivatives of y and u.
- It turns out that the derivative of the composition $f \circ g$ is the product of the derivative of f and the derivative of g.
- This important fact is called the Chain Rule.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

We will not prove this in class, but a proof can be found in the textbook.

Example (Chain Rule,)

Differentiate $\quad f(x) = \sqrt{x^2+1}.$

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Differentiate $f(x) = \sqrt{x^2 + 1}$.

Let $h(x) =$

Let $g(x) =$

Then $f(x) = g(h(x))$.

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$$\text{Differentiate} \quad f(x) = \sqrt{x^2+1}.$$
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Example (Chain Rule,)

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$$\text{Then} \quad f(x) = g(h(x)).$$

Example (Chain Rule,)

$$\begin{aligned} \text{Differentiate} \quad f(x) &= \sqrt{x^2+1}. \\ \text{Let} \quad h(x) &= x^2+1. \\ \text{Let} \quad g(x) &= \sqrt{x}. \\ \text{Then} \quad f(x) &= g(h(x)). \\ \text{Chain Rule:} \quad f'(x) &= g'(h(x))h'(x) \end{aligned}$$

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Differentiate $\quad y = \cos x^3.$

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& = \left(3u^2\right)(-\sin x) \\
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\end{aligned}$$

- In the example $y = \sin^2 x$, the outer function was a power function: $y = u^2$.

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- The derivative was $\frac{dy}{dx} = 2u\frac{du}{dx} = 2\sin x \cos x$.
- We can generalize this:

The Power Rule Combined with the Chain Rule
If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example (Chain Rule,)

Differentiate $\quad y = (x^3 - 1)^{100}$.

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Example (Chain Rule,)

$$\text{Differentiate} \quad y = (x^3 - 1)^{100}.$$

$$\text{Let} \quad u = x^3 - 1.$$

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Let $u = x^3 - 1$.

Then $y = u^{100}$.

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= \left(100u^{99}\right)\left(\quad\right)$$

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& = 300x^2(x^3 - 1)^{99}.
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example (Chain Rule,)

Differentiate $\quad f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}.$

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Find the derivative of

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9.$$

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$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{2t+1-2t+4}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Example

Find the derivative of $y = (2x + 1)^5(x^3 - x + 1)^4$.

Example

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Common factor $2(2x+1)^4(x^3 - x + 1)^3$:

$$= 2(2x+1)^4(x^3 - x + 1)^3(17x^3 + 6x^2 - 9x + 3)$$

Example (Chain Rule, general exponential function)

Differentiate $y = 2^x$.

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&= (e^{\ln 2})^x(\ln 2) \\
&= 2^x \ln 2.
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Theorem (The Derivative of a^x)

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

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$$= \frac{(\ln 10)10^x\cos\sqrt{10^x+1}}{2\sqrt{10^x+1}}.$$

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