Math 140
Lecture 15

Greg Maloney

with modifications by T. Milev

University of Massachusetts Boston

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Outline

1. (6.4) Derivatives of Logarithmic Functions
   - Logarithmic Differentiation
   - The Number *e* as a Limit

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   - Logarithmic Differentiation
   - The Number e as a Limit
2. (2.9) Linear Approximations and Differentials
   - Differentials

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Use the properties of logarithms to simplify.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for y'.

Note: If $f(x) < 0$, then we use $\ln|f(x)|$ instead as $\ln f(x)$ is not defined. We computed the derivative of $\ln|f(x)|$ in the previous lecture.

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The Number e as a Limit

Theorem (The Number e as a Limit)

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof.

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Let $f(x) = \ln x$. Then $f'(x) = 1/x$, so $f'(1) = 1$.

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Then use the fact that the exponential function is continuous:

$$e = e^1 =$$

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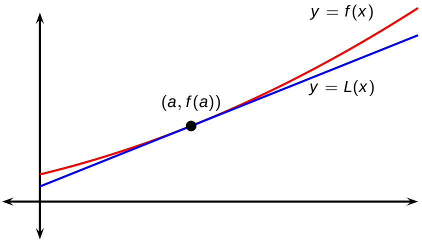
(3.9) Linear Approximations and Differentials

- Main idea: A curve is very close to its tangent line at the point of tangency.
- We can use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$.
- This approximation works well as long as x is near a.



Definition (Linearization of f at a)

The linear function whose graph is the tangent line at $(a, f(a))$ is called the linearization of f at a. Its equation is

$$L(x) = f(a) + f'(a)(x - a).$$

Definition (Linear Approximation of $f(x)$ near a)

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

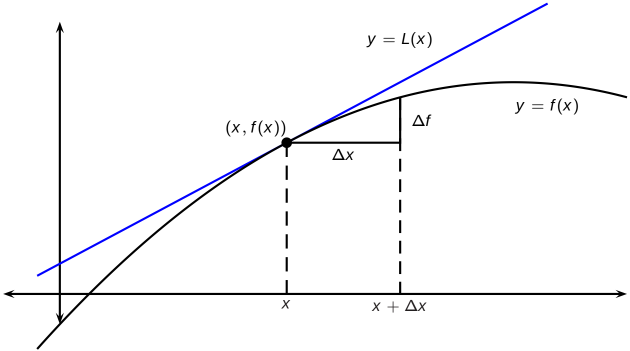
is called the linear approximation of f at a.

Let $y = f(x)$, $\Delta y := f(x) - f(a)$, and $\Delta x := x - a$.

Definition (Linear approx. $y = f(x)$ near a, alternative notation)

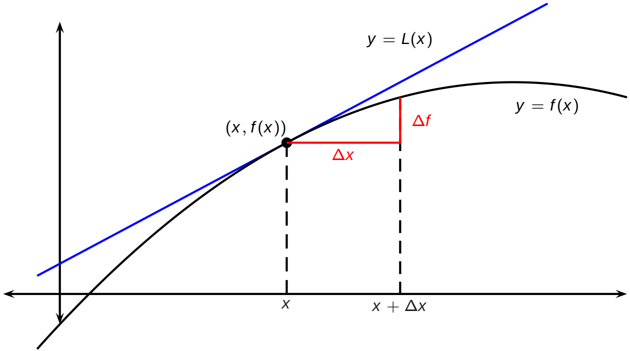
$$\Delta y \approx \frac{dy}{dx}\Delta x \quad .$$

Linear approximations



Function	f	L
Run	Δx	Δx
Rise	Δf	ΔL
Formula	$\Delta f = f(x + \Delta x) - f(x)$	$\Delta L = (\Delta x) f'(x)$

Linear approximations



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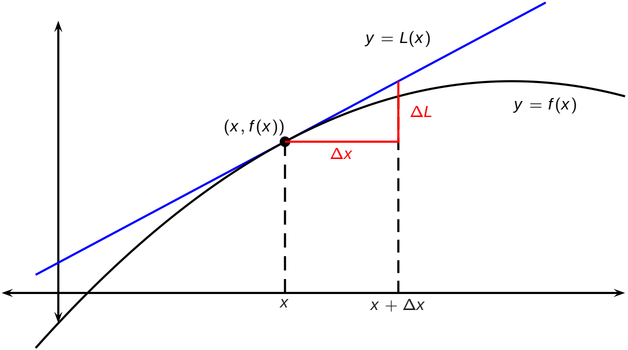

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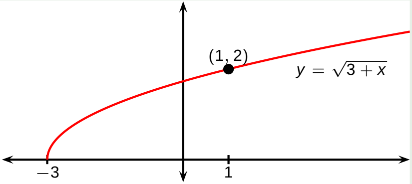
Example

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?



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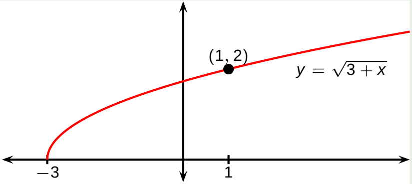

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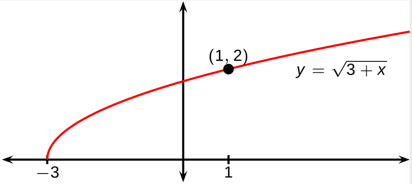

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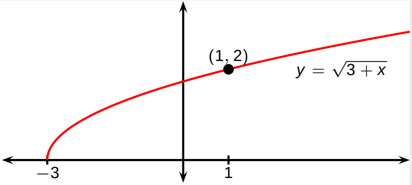

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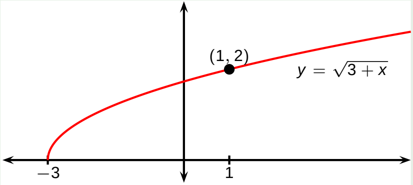

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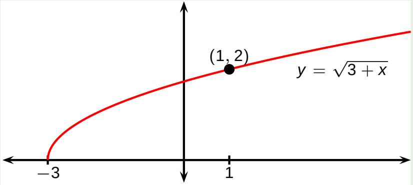

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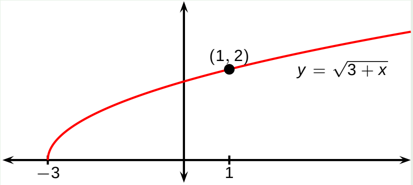

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- $f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$.
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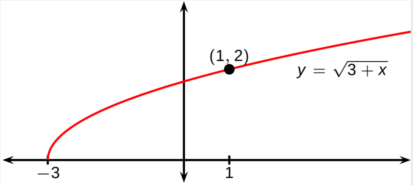

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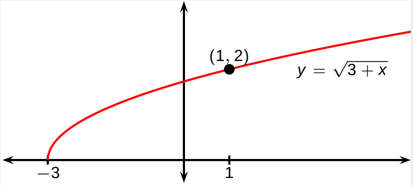

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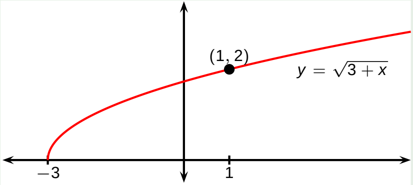

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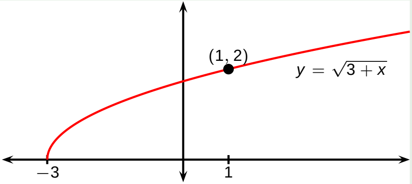

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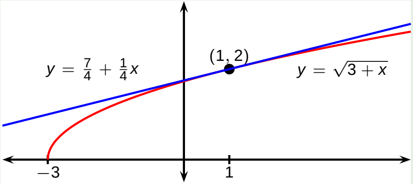

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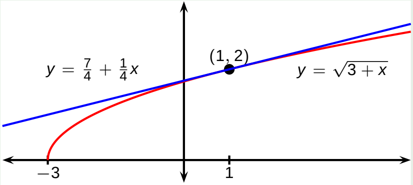

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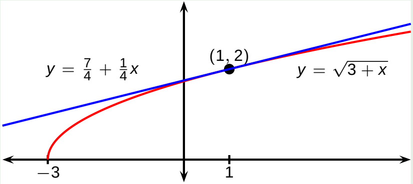

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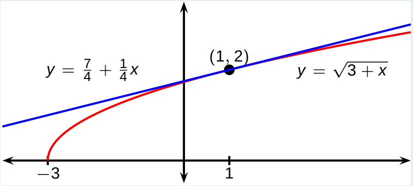

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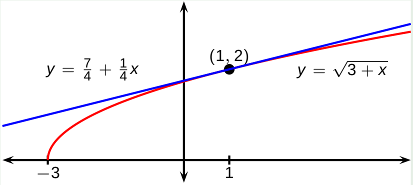

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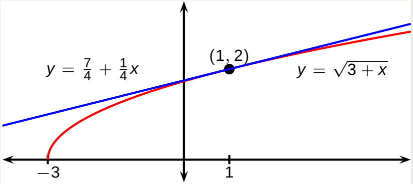

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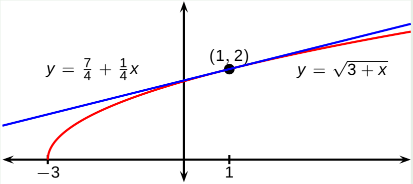

The linearization is above the curve, so these are overestimates.

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Compute Δy and $\Delta L = f'(x)\Delta x$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

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- Therefore $\Delta Ly = 0.7$, a pretty good approximation for $\Delta y = 0.717625.$