Math 140

Lecture 17

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with modifications by T. Milev

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Outline

1. (3.1) Maximum and Minimum Values
   - Fermat's Theorem
   - (3.1)The Closed Interval Method

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Fermat's Theorem

The next theorem gives a condition that can help to find local maxima and minima.

Theorem (Fermat's Theorem)

Let f be a function defined in an open interval around c and such that $f'(c)$ exists. If f has a local maximum or minimum at c, then $f'(c) = 0$.

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- Therefore $f'(c) \leq 0$ $\square$

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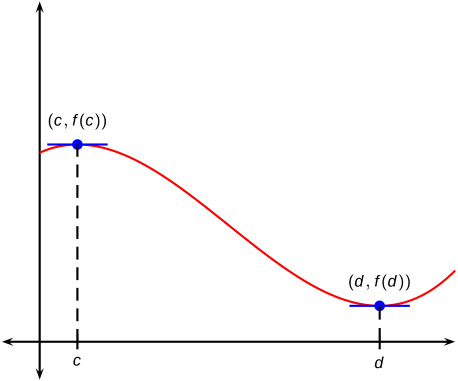

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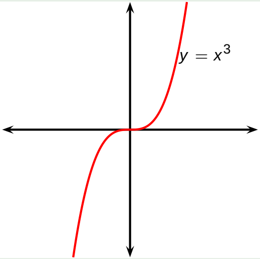

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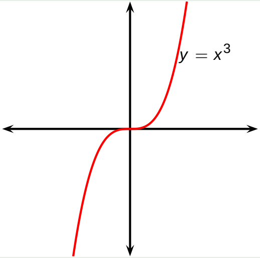

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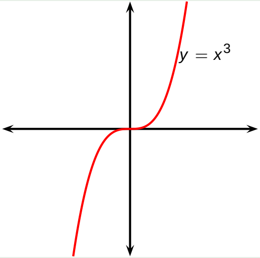

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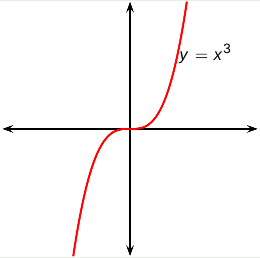

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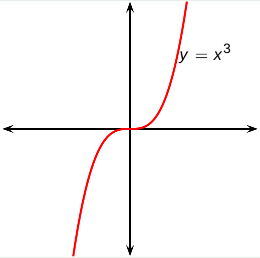

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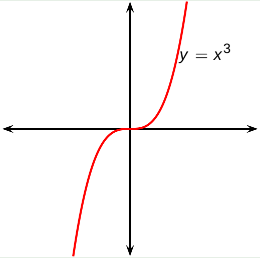

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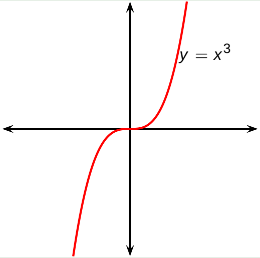

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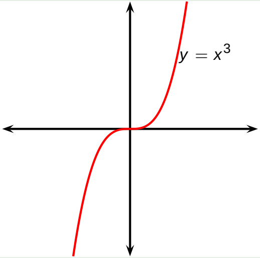

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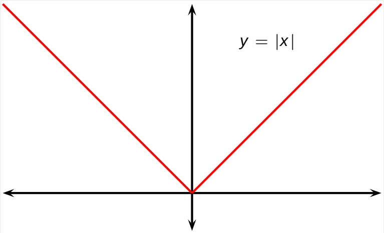

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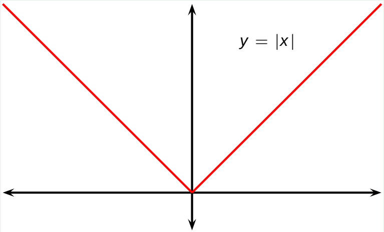

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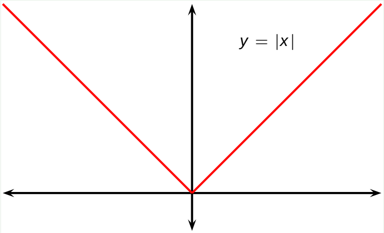

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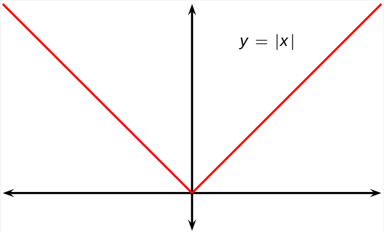

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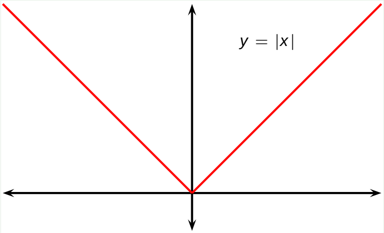

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Fermat's Theorem and Example 6 suggest that we should look at three types of points to find local maxima and minima:

1. Points c for which $f'(c) = 0$.
2. Points c for which $f'(c)$ doesn't exist.
3. Points c at end of intervals where f is defined. Here, we need also that f be defined at c.

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Fermat's Theorem says that if f has a local maximum or minimum at c, and c is not an endpoint, then c is a critical number for f.

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The Closed Interval Method

We know from the Extreme Value Theorem that a continuous function attains its absolute maximum and minimum on a closed interval $[a, b]$. The maximum might occur at an endpoint. The minimum might occur at an endpoint.

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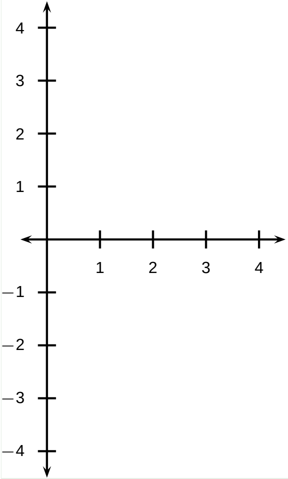
To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in $[a, b]$.
   - Find the values c with $f'(c) = 0$.
   - Find the values c where f' does not exist.
2. Find the values of f at the endpoints a and b.
3. The absolute maximum of f is maximum of the preceding values; the absolute minimum value is the minimum.

Example

Find the absolute maximum and minimum values of the function $f(x) = -x^3 + 2x^2 + 4x - 5$ on the interval $[1, 3]$.



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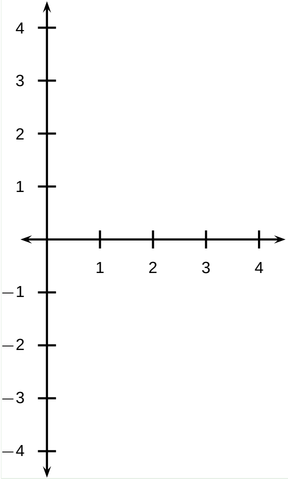

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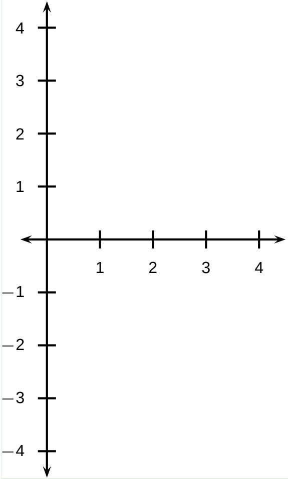

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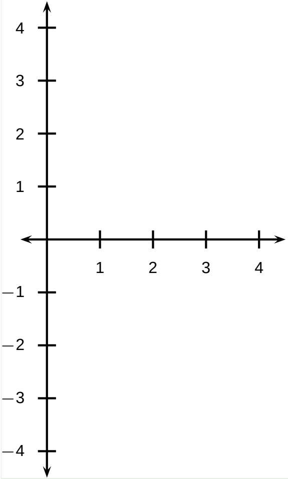

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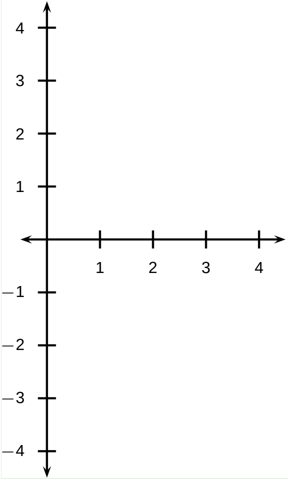

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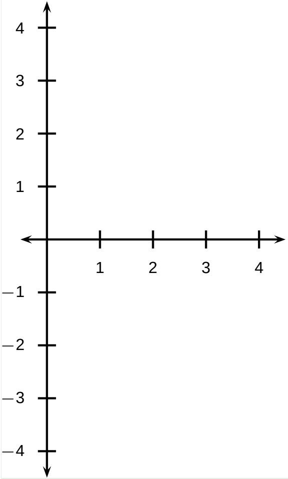

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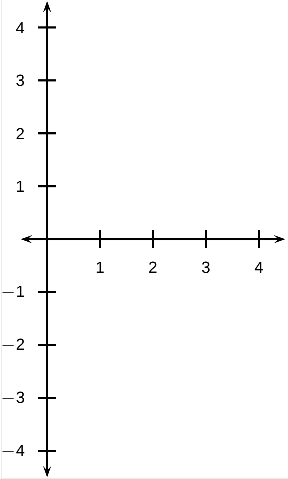

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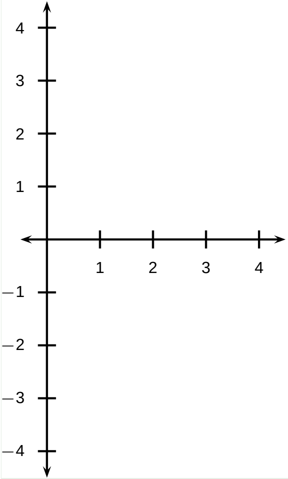

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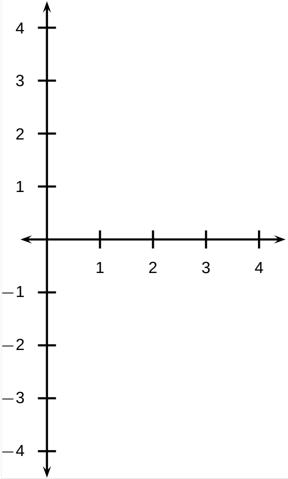

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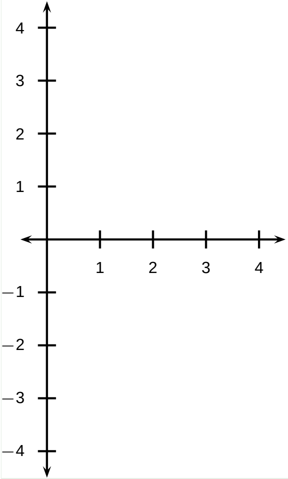

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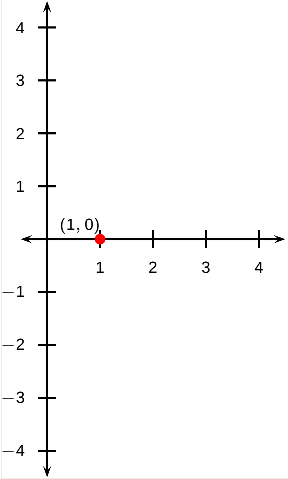

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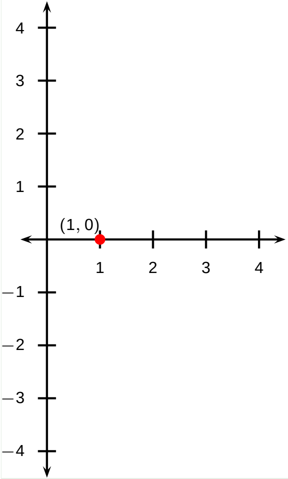

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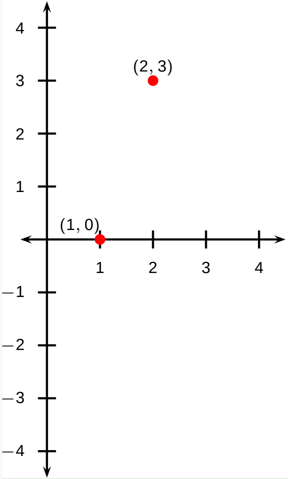

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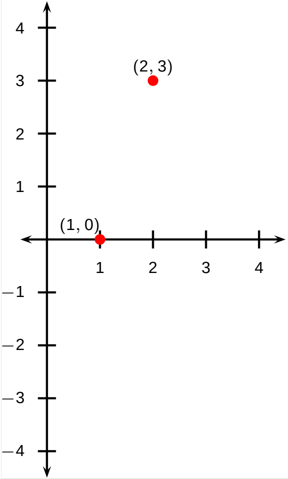

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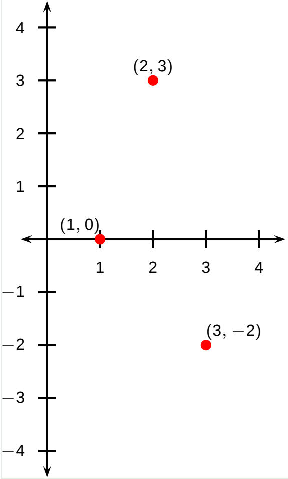

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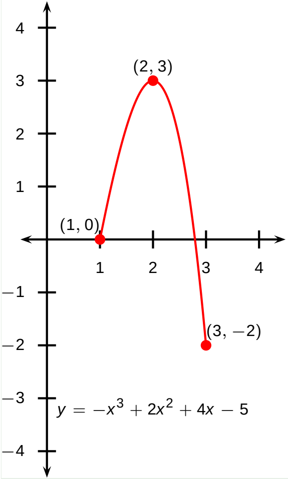

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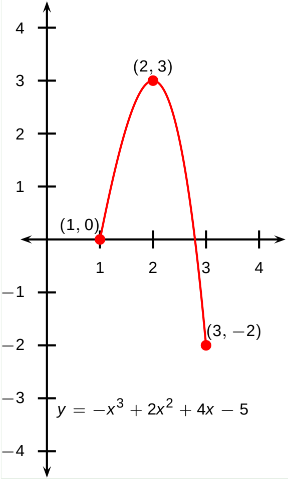

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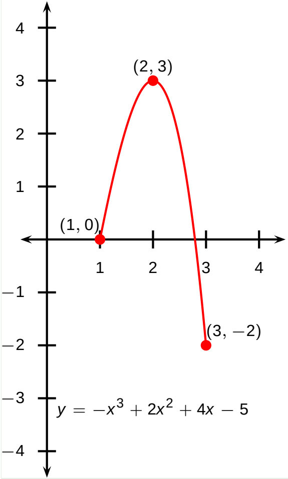

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Absolute maximum: 3. Absolute minimum:

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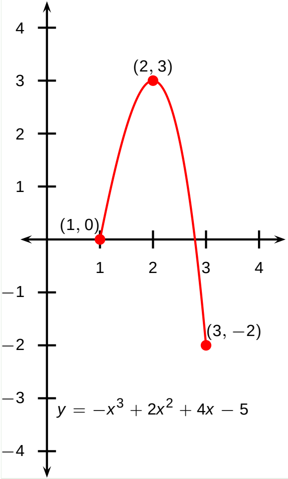

$$f'(x) = -3x^2 + 4x + 4$$
$$= (-3x - 2)(x - 2)$$

If $f'(x) = 0$, $x = -\frac{2}{3}$ or 2.

Need to check:

1. The critical numbers of f in $[a, b]$.
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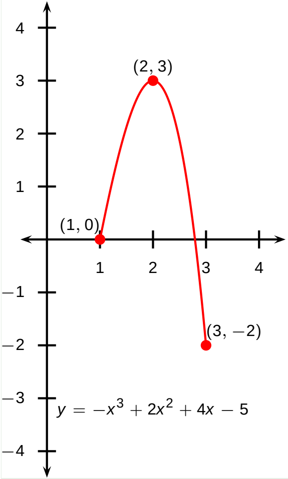

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The Mean Value Theorem

- Many results in this section (and others) depend on an important theorem, called the Mean Value Theorem.
- Before we can prove the Mean Value Theorem, we need to prove Rolle's Theorem.

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three conditions:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

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The proof breaks down into three cases:

1. f is a horizontal line.
2. $f(x) > f(a)$ for some x in (a, b).
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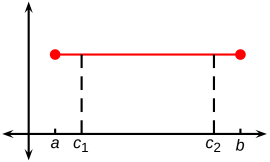

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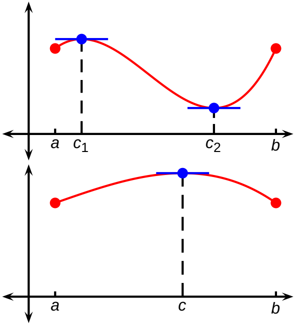

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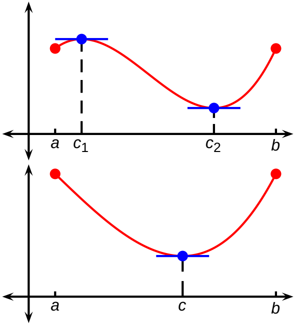

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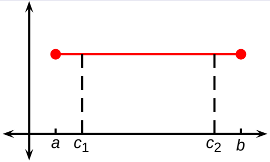

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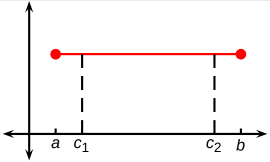

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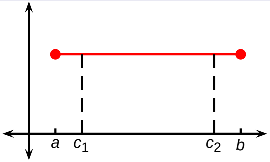

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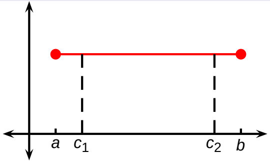

1. f is a horizontal line.
   - Then $f'(x) = 0$.
   - Therefore we can take c to be any number in (a, b). □

Theorem (Rolle's Theorem)

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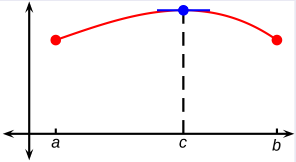

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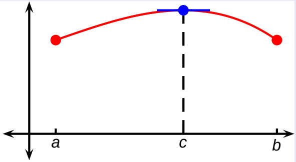

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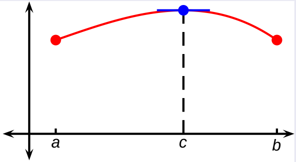

2. $f(x) > f(a)$ for some x in (a, b).
   - By the Extreme Value Theorem, f has a maximum in $[a, b]$.
   - Since $f(x) > f(a)$, this value is attained at some c in (a, b).

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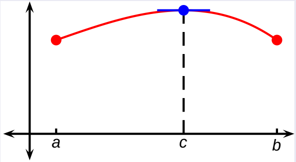

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- Since $f(x) > f(a)$, this value is attained at some c in (a, b).
- Fermat's Theorem: $f'(c) = 0$. $\square$

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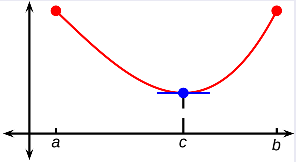

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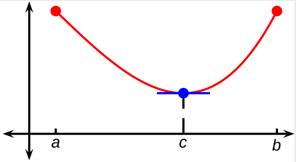

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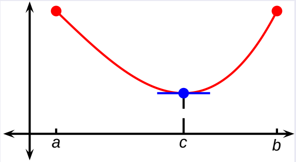

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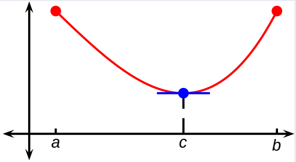

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