Exam II
Math 140 Calculus I
Instructor: Todor Milev

Name:

Problem	1	2	3	4	5	6	7	8	9	10	$\sum$
Score											

The exam is closed books, no calculators allowed. 100 points are worth 100% of the grade.

Problem 1 *(10 pts) Define what it means for a function to be differentiable at a point. Define derivative of a function.*

Solution. By definition, a function f is differentiable at a point x if the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=0$ exists. If this the case, the latter limit is called the derivative of f at x.

Problem 2 *(10 pts) Compute the derivative of the function. Simplify your answer to a single fraction.*

$$f(x)=\frac{1-2x}{1+\frac{3}{x}}$$

Solution.

$$\begin{aligned}\left(\frac{1-2x}{1+\frac{3}{x}}\right)' &= \frac{(1-2x)'(1+\frac{3}{x})-(1-2x)(1+3x^{-1})'}{\left(1+\frac{3}{x}\right)^2}\\ &= \frac{-2(1+3x^{-1})-(1-2x)(-3)x^{-2}}{x^{-2}(x+3)^2}\\ &= \frac{-2-6x^{-1}+3x^{-2}-6x^{-1}}{x^{-2}(x+3)^2} = \frac{\cancel{x^{-2}}(-2x^2-12x+3)}{\cancel{x^{-2}}(x+3)^2}\\ &= \frac{-2x^2-12x+3}{(x+3)^2}\end{aligned}$$

Problem 3 *(10 pts)Compute the derivative of the function.*

$$f(x) = 2^{3^{(\ln x)}} \quad .$$

Solution. We studied in class that $(p^x)' = p^x \ln p$. Therefore we can apply consecutively the chain rule to get

$$\left(2^{3^{(\ln x)}}\right)' = 2^{3^{(\ln x)}} (\ln 2) \left(3^{(\ln x)}\right)' = 2^{3^{(\ln x)}} (\ln 2) 3^{\ln x} (\ln 3) \underbrace{(\ln x)'}_{=\frac{1}{x}} = \frac{\ln 2 \ln 3}{x} 2^{3^{(\ln x)}} 3^{\ln x} \quad .$$

Problem 4 *(10 pts)Compute the derivative of the function.*

$$f(x) = \sec^3 \left(\frac{2}{x}\right).$$

Solution. As studied in class, $(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{-(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$. Therefore

$$\sec^3 \left(\frac{2}{x}\right)' = 3 \sec^2 \left(\frac{2}{x}\right) \left(\sec \left(\frac{2}{x}\right)\right)' = 3 \sec^2 \left(\frac{2}{x}\right) \sec \left(\frac{2}{x}\right) \tan \left(\frac{2}{x}\right) \left(\frac{2}{x}\right)' = -\frac{6}{x^2} \sec^3 \left(\frac{2}{x}\right) \tan \left(\frac{2}{x}\right) \quad .$$

Problem 5 *(15 pts)Compute the limit.*

$$\lim_{x\to 0} \frac{x^2}{\cos(8x)-1} \quad .$$

Solution. We have that $\cos(8x) = 1 - 2\sin^2(4x)$. As studied in class, $\lim_{y\to 0} \frac{\sin y}{y} = 1$ and therefore

$$\lim_{x\to 0} \frac{x^2}{\cos(8x)-1} = \lim_{x\to 0} \frac{x^2}{\cancel{1} - 2\sin^2(4x) - \cancel{1}} = \lim_{x\to 0} \frac{\frac{(4x)^2}{16}}{-2\sin^2(4x)} = -\frac{1}{32} \lim_{x\to 0} \frac{(4x)^2}{\sin^2(4x)} = -\frac{1}{32} \lim_{x\to 0} \frac{1}{\frac{\sin^2(4x)}{(4x)^2}} = -\frac{1}{32} \quad .$$

Problem 6 *(10 pts) Find the equation of the tangent line to the function*

$$y = x(\ln x)$$

at the point $(1, 0)$.

Solution.

$$\frac{dy}{dx} = (x \ln x)' = (x)' \ln x + x(\ln x)' = \ln x + x\frac{1}{x} = \ln x + 1 \quad .$$

Therefore $\frac{dy}{dx}_{|x=1} = \ln 1 + 1 = 1$. Therefore the equation of the tangent line to the graph of $y = x(\ln x)$ at $(1, 0)$ is $(y - 0) = 1 \times (x - 1)$, or in other words, $y = x - 1$.

Problem 7 *(10 pts) Use implicit differentiation to express* $\frac{dy}{dx}$ *via* y *and* x*, where* x *and* y *satisfy the following relation.* $x^4(x-y) = y^2(3x+y)$.

Solution.

$$\begin{array}{rcll} x^4(x-y) &=& y^2(3x+y) & \\ x^5 - x^4 y &=& 3y^2 x + y^3 & \frac{d}{dx} \\ 5x^4 - 4x^3 y - x^4 \frac{dy}{dx} &=& 3y^2 + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} & \\ \frac{dy}{dx}\left(x^4 + 6xy + 3y^2\right) &=& 5x^4 - 4x^3 y - 3y^2 & \\ \frac{dy}{dx} &=& \frac{5x^4-4x^3y-3y^2}{x^4+6xy+3y^2} & . \end{array}$$

Problem 8 *(10 pts) Use implicit differentiation to find an equation of the tangent line to the curve*

$$x^5 + x^3 y - y^6 = -1$$

at the point $(-1,-1)$.

Solution. Direct substitution shows that $(-1,-1)$ satisfies the above equation.

$$\begin{array}{rcll} x^5 + x^3 y - y^6 &=& -1 & \frac{d}{dx} \\ 5x^4 + 3x^2 y + x^3\frac{dy}{dx} - 6y^5\frac{dy}{dx} &=& 0 & \end{array}$$

We can substitute $x=-1, y=-1$ in the above expression to get

$$\begin{array}{rcll} 5(-1)^4 + 3(-1)^2(-1) + (-1)^3\frac{dy}{dx}_{|x=-1} - 6(-1)^5\frac{dy}{dx}_{|x=-1} &=& 0 & \\ 5 - 3 - \frac{dy}{dx}_{|x=-1} + 6\frac{dy}{dx}_{|x=-1} &=& 0 & \\ \frac{dy}{dx}_{|x=-1} &=& -\frac{2}{5} & . \end{array}$$

Therefore the equation of the tangent line at the point $(-1,-1)$ is $y-(-1) = -\frac{2}{5}(x-(-1))$ or $y = -\frac{2}{5}x - \frac{7}{5}$.

Problem 9 *(15 pts) Compute* $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x$.

Solution. As studied in class, $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$. Set $t = -\frac{3}{x}$. Therefore as $x \to \infty$, $t \to 0^-$, and we have

$$\lim_{\substack{x\to\infty\\ t=-\frac{3}{x}\\ t\to 0^-}}\left(1-\frac{3}{x}\right)^x = \lim_{t\to 0^-}(1+t)^{-\frac{3}{t}} = \lim_{t\to 0^-}\left((1+t)^{\frac{1}{t}}\right)^{-3} = \left(\lim_{t\to 0^-}(1+t)^{\frac{1}{t}}\right)^{-3} = e^{-3} = \frac{1}{e^3} \quad .$$

The problems before this one sum up to 100%.

Problem 10 *(10 pts) For a positive integer* n*, prove the power rule* $\frac{d}{dx}(x^n) = nx^{n-1}$ *using the definition of derivative.*

Solution. By the definition of derivative we have

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{(\cancel{x}+h-\cancel{x})((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1})}{h}\\ &= \lim_{h\to 0}\left(\underbrace{(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}}_{n\text{ summands total}}\right) = nx^{n-1} \quad .\end{aligned}$$