# Math 140

## Lecture 23

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# Outline

1 (5.4) The Fundamental Theorem of Calculus

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1. (5.4) The Fundamental Theorem of Calculus
2. (5.5) The Substitution Rule
   - Definite Integrals

# The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus establishes a connection between differential calculus and integral calculus.
- It allows us to compute integrals very easily, without finding limits of Riemann sums.
- It has two different parts.
- Part 1 says, roughly, that "differentiation undoes integration."
- Part 2 says, roughly, that "integration undoes differentiation."

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- It has two different parts.
- Part 1 says, roughly, that "differentiation undoes integration."
- Part 2 says, roughly, that "integration undoes differentiation."
- Part 1 deals with functions of the form
$$g(x) = \int_a^x f(t)\mathrm{d}t$$
where $f$ is a continuous function on $[a, b]$ and $x$ varies between $a$ and $b$.

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- If we let $x$ vary, then $\int_a^x f(t)\mathrm{d}t$ varies.
- If $f$ is positive, then $g$ can be interpreted as the area under $f$ from $a$ to $x$.

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**Theorem (The Fundamental Theorem of Calculus, Part 1)**

*If $f$ is continuous on $[a, b]$, then the function $g$ defined by*

$$g(x) = \int_a^x f(t)dt$$

*is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = f(x)$.*

## Example (Example 3, p. 370)

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For each formula $g(x)$, find the derivative $g'(x)$.

| $g(x)$ | $g'(x)$ |
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| $\int_0^x \sin(t^2+1)\cos(t^3+2)\mathrm{d}t$ | |
| $\int_{35}^x \frac{1+r^2+4r^3}{1-r^4}\mathrm{d}r$ | |
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## Example (Chain Rule, FTC Part 1, Example 5, p. 370)

$$\text{Differentiate} \quad y = \int_0^{x^4} \sec t \, dt.$$

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$$\begin{aligned} \text{Differentiate} \quad y &= \int_0^{x^4} \sec t \, dt. \\ \text{Let} \quad u &= x^4. \\ \text{Then} \quad y &= \int_0^{u} \sec t \, dt. \end{aligned}$$

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Differentiate $y = \int_0^{x^4} \sec t \, dt.$

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Then $y = \int_0^{u} \sec t \, dt.$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

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$$= (\sec u)\left(4x^3\right)$$

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$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\sec u)\left(4x^3\right) \\ &= 4x^3 \sec(x^4). \end{aligned}$$

### Theorem (The Fundamental Theorem of Calculus)

*Suppose f is continuous on $[a, b]$. Then*

1. *If* $g(x) = \int_a^x f(t)dt$*, then* $g'(x) = f(x)$.
2. $\int_a^b f(x)dx = F(b) - F(a)$*, where F is any antiderivative of f.*

Part 2 of the FTC is the Evaluation Theorem, which we learned in Section 5.3.

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This approach works more generally, and is called the Substitution Rule.

**Theorem (The Substitution Rule)**

*If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

This is the opposite of the Chain Rule.

## Example (Substitution Rule, Example 1, p. 376)

Find $$\int x^3 \cos(x^4 + 2) \mathrm{d}x.$$

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$$x^3 \mathrm{d}x = \frac{1}{4} \mathrm{d}u.$$

## Example (Substitution Rule, Example 1, p. 376)

Find $\displaystyle\int x^3\cos(x^4+2)\mathrm{d}x.$

$$\text{Let}\quad u = x^4+2.$$

$$\text{Then}\quad \mathrm{d}u = 4x^3\mathrm{d}x$$

$$x^3\mathrm{d}x = \frac{1}{4}\mathrm{d}u.$$

Substitute: $\displaystyle\int x^3\cos(x^4+2)\mathrm{d}x = \int \quad \cos u$

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## Example (Substitution Rule, Example 1, p. 376)

Find $\displaystyle\int x^3 \cos(x^4+2) \mathrm{d}x.$

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$$\text{Substitute:} \quad \int x^3 \cos(x^4+2) \mathrm{d}x = \int \frac{1}{4} \cos u \mathrm{d}u$$

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## Example (Substitution Rule, Example 1, p. 376)

Find $\displaystyle\int x^3\cos(x^4+2)\mathrm{d}x.$

$$\begin{aligned}\text{Let}\quad u &= x^4+2.\\ \text{Then}\quad \mathrm{d}u &= 4x^3\mathrm{d}x\\ x^3\mathrm{d}x &= \frac{1}{4}\mathrm{d}u.\end{aligned}$$

$$\text{Substitute:}\quad \int x^3\cos(x^4+2)\mathrm{d}x = \int \frac{1}{4}\cos u\mathrm{d}u = \frac{1}{4}\sin u$$

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Find $\displaystyle\int x^3 \cos(x^4+2)\mathrm{d}x.$

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$$= \frac{1}{4}\sin u + C$$

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\text{Substitute:}\quad \int x^3\cos(x^4+2)\mathrm{d}x &= \int \frac{1}{4}\cos u\mathrm{d}u\\
&= \frac{1}{4}\sin u + C\\
&= \frac{1}{4}\sin(x^4+2)+C.
\end{aligned}$$

## Example (Substitution Rule, Example 2, p. 377)

Find $\displaystyle\int \sqrt{2x+1}\,dx.$

## Example (Substitution Rule, Example 2, p. 377)

Find $$\int \sqrt{2x+1}\,\mathrm{d}x.$$

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$$\text{Find} \quad \int \sqrt{2x+1}\,\mathrm{d}x.$$

$$\text{Let} \quad u = 2x + 1.$$

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## Example (Substitution Rule, Example 2, p. 377)

Find $\displaystyle\int \sqrt{2x+1}\,\mathrm{d}x$.

Let $u = 2x + 1$.

Then $\mathrm{d}u = 2\mathrm{d}x$

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Find $\int \sqrt{2x+1}dx$.

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Find $\displaystyle\int \sqrt{2x+1}\,\mathrm{d}x.$

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Substitute: $\displaystyle\int \sqrt{2x+1}\,dx = \int \sqrt{u}$

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Find $\displaystyle\int \sqrt{2x+1}\mathrm{d}x.$

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Substitute: $\displaystyle\int \sqrt{2x+1}\mathrm{d}x = \int \frac{1}{2}\sqrt{u}\mathrm{d}u$

$=$

## Example (Substitution Rule, Example 2, p. 377)

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$$\begin{aligned} \text{Let} \quad u &= 2x+1. \\ \text{Then} \quad du &= 2dx \\ dx &= \frac{1}{2}du. \end{aligned}$$

Substitute: $\displaystyle\int \sqrt{2x+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du$

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}$$

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Find $\displaystyle\int \sqrt{2x+1}\mathrm{d}x.$

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$$\begin{aligned} \text{Substitute:} \quad \int \sqrt{2x+1}\mathrm{d}x &= \int \frac{1}{2}\sqrt{u}\mathrm{d}u \\ &= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C \end{aligned}$$

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Find $\int \sqrt{2x+1}\mathrm{d}x.$

$$\text{Let} \quad u = 2x + 1.$$

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$$\text{Substitute:} \quad \int \sqrt{2x+1}\mathrm{d}x = \int \frac{1}{2}\sqrt{u}\mathrm{d}u$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C.$$

## Example (Substitution Rule, Example 3, p. 377)

Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x.$

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Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x.$

Let $u = 1 - 4x^2.$

## Example (Substitution Rule, Example 3, p. 377)

$$\text{Find} \quad \int \frac{x}{\sqrt{1-4x^2}} dx.$$

$$\text{Let} \quad u = 1 - 4x^2.$$

$$\text{Then} \quad du =$$

## Example (Substitution Rule, Example 3, p. 377)

Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x.$

Let $u = 1 - 4x^2.$

Then $\mathrm{d}u = -8x\mathrm{d}x$

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Substitute: $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x = \int \quad \frac{1}{\sqrt{u}}$

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Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x.$

$$\begin{aligned} \text{Let}\quad u &= 1-4x^2. \\ \text{Then}\quad \mathrm{d}u &= -8x\mathrm{d}x \\ x\mathrm{d}x &= -\frac{1}{8}\mathrm{d}u. \end{aligned}$$

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Let $u = 1 - 4x^2.$

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Substitute:
$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x &= \int -\frac{1}{8}\frac{1}{\sqrt{u}}\mathrm{d}u\\ &= -\frac{1}{8}\cdot\frac{u^{1/2}}{1/2} + C\\ &= -\frac{1}{4}\sqrt{1-4x^2} + C.\end{aligned}$$

## Example (Substitution Rule, Example 4, p. 378)

Find $$\int e^{5x} \mathrm{d}x.$$

## Example (Substitution Rule, Example 4, p. 378)

$$\text{Find} \quad \int e^{5x} \mathrm{d}x.$$

$$\text{Let} \quad u =$$

## Example (Substitution Rule, Example 4, p. 378)

Find $\displaystyle\int e^{5x} dx.$

Let $u = 5x.$

## Example (Substitution Rule, Example 4, p. 378)

$$\text{Find} \quad \int e^{5x} \mathrm{d}x.$$

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$$\text{Find} \quad \int e^{5x} \mathrm{d}x.$$

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$$\text{Let} \quad u = 5x.$$

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\text{Substitute:} \quad \int e^{5x}\mathrm{d}x &= \int \frac{1}{5} e^{u} \mathrm{d}u \\
&= \frac{1}{5} e^{u} + C
\end{aligned}$$

## Example (Substitution Rule, Example 4, p. 378)

Find $\displaystyle\int e^{5x} \mathrm{d}x.$

$$\text{Let} \quad u = 5x.$$

$$\begin{aligned} \text{Then} \quad \mathrm{d}u &= 5\mathrm{d}x \\ \mathrm{d}x &= \frac{1}{5}\mathrm{d}u. \end{aligned}$$

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# Definite Integrals

There are two ways to find a definite integral with the Substitution Rule:

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2. Change the limits of integration when the variable is changed.

**Theorem (The Substitution Rule for Definite Integrals)**

*If $u = g(x)$ and $g$ and $f$ are continuous, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

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Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

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Find $\int_0^4 \sqrt{2x+1}\, dx$.

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Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
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## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x =$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
- When $x = 0$, $u =$
- When $x = 4$, $u =$

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$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \int \quad \sqrt{\phantom{x}}$$

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$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \int \qquad \sqrt{u}$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,dx$.

- Let $u = 2x + 1$.
- Then $du = 2dx$.
- Therefore $dx = \frac{1}{2}du$.
- When $x = 0$, $u = 1$.
- When $x = 4$, $u = 9$.

$$\int_0^4 \sqrt{2x+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

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- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
- When $x = 0$, $u = 1$.
- When $x = 4$, $u = 9$.

$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \int_1 \frac{1}{2}\sqrt{u}\,\mathrm{d}u$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
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- When $x = 0$, $u = 1$.
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$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \int_1^9 \frac{1}{2}\sqrt{u}\,\mathrm{d}u$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
- When $x = 0$, $u = 1$.
- When $x = 4$, $u = 9$.

$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \int_1^9 \frac{1}{2}\sqrt{u}\,\mathrm{d}u$$

$$= \left[\frac{1}{2}\cdot\frac{2}{3}(u)^{3/2}\right]_1^9$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\, dx$.

- Let $u = 2x + 1$.
- Then $du = 2dx$.
- Therefore $dx = \frac{1}{2}du$.
- When $x = 0$, $u = 1$.
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$$\begin{aligned}\int_0^4 \sqrt{2x+1}\, dx &= \int_1^9 \frac{1}{2}\sqrt{u}\, du \\ &= \left[\frac{1}{2}\cdot\frac{2}{3}(u)^{3/2}\right]_1^9 \\ &= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2}\end{aligned}$$

## Example (Example 6, p. 379)

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
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$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,\mathrm{d}x &= \int_1^9 \frac{1}{2}\sqrt{u}\,\mathrm{d}u \\ &= \left[\frac{1}{2}\cdot\frac{2}{3}(u)^{3/2}\right]_1^9 \\ &= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{1}{3}(27-1) = \frac{26}{3}\end{aligned}$$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2}$.

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- Let $u =$
- Then $\mathrm{d}u =$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u =$
- Then $du =$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
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## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
- Then $du = -5\,dx$.

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Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
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## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
- Then $du = -5\,dx$.
- Therefore $dx = -\frac{1}{5}du$.

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
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- When $x = 1$, $u =$
- When $x = 2$, $u =$

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$$\int_1^2 \frac{dx}{(3-5x)^2} = \int \qquad \frac{}{u^2}$$

## Example (Example 7, p. 379)

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$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int \qquad \frac{du}{u^2}$$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
- Then $\mathrm{d}u = -5\,\mathrm{d}x$.
- Therefore $\mathrm{d}x = -\frac{1}{5}\mathrm{d}u$.
- When $x = 1$, $u = -2$.
- When $x = 2$, $u = -7$.

$$\int_1^2 \frac{\mathrm{d}x}{(3-5x)^2} = -\frac{1}{5}\int_{-2} \quad \frac{\mathrm{d}u}{u^2}$$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
- Then $du = -5\,dx$.
- Therefore $dx = -\frac{1}{5}du$.
- When $x = 1$, $u = -2$.
- When $x = 2$, $u = -7$.

$$\int_1^{2} \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

- Let $u = 3 - 5x$.
- Then $du = -5\,dx$.
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$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7}\frac{du}{u^2}$$

$$= -\frac{1}{5}\cdot\left[-\frac{1}{u}\right]_{-2}^{-7}$$

## Example (Example 7, p. 379)

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$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$
$$= -\frac{1}{5}\cdot\left[-\frac{1}{u}\right]_{-2}^{-7}$$
$$= \left[\frac{1}{5u}\right]_{-2}^{-7}$$

## Example (Example 7, p. 379)

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$$\begin{aligned}\int_1^2 \frac{dx}{(3-5x)^2} &= -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}\\ &= -\frac{1}{5}\cdot\left[-\frac{1}{u}\right]_{-2}^{-7}\\ &= \left[\frac{1}{5u}\right]_{-2}^{-7} = \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right)\end{aligned}$$

## Example (Example 7, p. 379)

Find $\int_1^2 \frac{dx}{(3-5x)^2}$.

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