**Exercise 1** *(trapezoidal method for advection)*

Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{Ex1a}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix $A$? Don't forget the boundary conditions.

(b) Suppose we want to fix the Courant number $ak/h$ as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix $A$ from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (Ex1a). What is the amplification factor $g(\xi)$?

(d) For what range of $ak/h$ will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Suppose we use the same method (Ex1a) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (Ex1a) can be applied at $x_{m+1}$). For what range of $ak/h$ will the CFL condition be satisfied in this case? What are the eigenvalues of the $A$ matrix for this case and when will the method be stable?

**Exercise 2** *(computing with leapfrog)*

(a) Modify the code in the Lax-Wendroff worksheet to implement the leapfrog method with periodic boundary conditions. Verify that it is second order accurate by reducing $k$ and $h$ while keeping the CFL number $ka/h$ fixed, and comparing the error (using the grid-norm version of the 1-norm) for different mesh widths.

Note that you will have to specify two levels of initial data. For the convergence test set $U_j^1 = u(x_j, k)$, the true solution at time $k$.

For initial data use a wave packet

$$\eta(x) = \exp(-\beta(x-0.5)^2)\cos(\xi x) \tag{Ex2a}$$

on the domain $0 \le x \le 1$ with $\xi = 80$, $a = 1$ and final time $T = 1$, so that the correct solution is just the initial condition.

(b) Using $\beta = 100$, $\xi = 150$ and $U_j^1 = u(x_j, k)$, estimate the group velocity of the wave packet computed with leapfrog using $m = 199$ and $k = 0.4h$. How well does this compare with the value (10.52) predicted by the modified equation?