Analysis of Boolean Functions April 4, 2013

Lecture 7: Relationship between Influence and Structure

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In this lecture we are discussing what can we learn about the structure/complexity of the boolean function from its influence, and also, what can we learn about the influence from the structure.

For most of the proofs in this lecture, we assume the function $f : \{-1,1\}^n \to \{-1,1\}$ is balanced, i.e. $\Pr_x[f(x) = 1] = \Pr_x[f(x) = -1] = \frac{1}{2}$. A few comments would be made when extending the result to unbalanced cases.

1 Recap

The influence of a boolean function has a nice implication of the structure. Recall the definition of influence:

Definition 1. *The influence of the variable x_i on f is*

$$\mathrm{Inf}_i(f) = \Pr_x[f(x) \neq f(x^{\oplus i})]$$

where $x^{\oplus i}$ is x with the i-th bit flipped.

Definition 2. *The total influence of f is*

$$\mathrm{Inf}(f) = \sum_i \mathrm{Inf}_i(f)$$

Some facts about the influence:

1. $\mathrm{Inf}_i(f) \geq |\hat{f}_{S=\{i\}}(S)| = |\hat{f}(i)|$. Since

$$\mathrm{Inf}_i(f) = \Pr_x[f(x) = 1 \wedge f(x^{\oplus i}) = -1] + \Pr_x[f(x) = -1 \wedge f(x^{\oplus i}) = 1]$$

while

$$\hat{f}(i) = \mathbb{E}[f(x)\chi_i(x)] = \Pr_x[f(x) = 1 \wedge f(x^{\oplus i}) = -1] - \Pr_x[f(x) = -1 \wedge f(x^{\oplus i}) = 1]$$

2. $\mathrm{Inf}(f) = \sum_{S \subseteq [n]} |S| \hat{f}(S)^2$

We've learned the (total) influences on some specific functions:

- Dictator function has the smallest influence among all balanced functions: $\mathrm{Inf}(\chi_i) = 1$
- Parity has the largest: $\mathrm{Inf}(X_{[n]}) = n$

- $\mathrm{Inf}(\mathsf{Majority}) = \Theta(\sqrt{n})$
- $\mathrm{Inf}(\mathsf{Tribes}) = \Theta(\log(n))$

To minimize the largest influence among all variables, Tribes achieves the best lower bound:

Theorem 3 (KKL)**.** *Let $f : \{-1,1\}^n \to \{-1,1\}$ be a balanced function, there exists $i \in [n]$ such that $\mathrm{Inf}_i(f) = \Omega(\log n/n)$.*

Also recall the definition of Junta:

Definition 4 (Junta)**.** *Let $f : \{-1,1\}^n \to \{-1,1\}$. Then f is a Junta on J iff there exists some function $g : \{-1,1\}^{|J|} \to \{-1,1\}$ such that $f(x_1, \ldots, x_n) = g(x_J)$.*

In other words, a *k-Junta* is a boolean function of n variables that depends only on an unknown subset of k variables. Usually, when we say f is a Junta, it means that f is a k-Junta for some $k = O(1)$.

If a boolean function is very close to Junta, then its influence is very small. The other direction also holds, but is non-trivial. However, we've already proved it in Friedgut's Junta Theorem:

Theorem 5 (Friedgut's Junta Theorem)**.** *Let $f : \{-1,1\}^n \to \{-1,1\}$. Then for every $0 < \epsilon < 1$, there exists some Junta g on J with $|J| = 2^{O(Inf(f)/\epsilon)}$ such that $\|f - g\|_2 \leq \epsilon$.*

2 From Influence to Structure

Can we learn something about the structure/complexity of the boolean function, if we know the influence of the boolean function f is very close to 1, say, $1 + \epsilon$? The FKN corollary rules out the connection from the influence to the structure:

Corollary 6 (Friedgut-Kalai-Naor)**.** *Let $f : \{-1,1\}^n \to \{-1,1\}$ be a balanced function. If $\mathrm{Inf}(f) \leq 1 + \epsilon$, then f is 2ϵ-close to dictator function (or anti-dictator function).*

We've already proven the following FKN theorem in the previous lectures.

Theorem 7 (Friedgut-Kalai-Naor[3])**.** *If $\sum_{|S|=1} \hat{f}(S)^2 \geq 1-\epsilon$, then $\|f - \chi_i\|_2 \leq O(\epsilon)$ or $\|f + \chi_i\|_2 \leq O(\epsilon)$.*

A recent, unpublished proof by Eric Blais, Li-Yang Tan and Andrew Wan shows that the FKN corollary has a more elementary proof:

Proof of FKN Corollary. We start by the following claims and observations:

Claim 8. *If $\mathrm{Inf}(f) \leq 1 + \epsilon$, then $\sum_{|S|=1} \hat{f}(S)^2 \geq 1 - \epsilon$*

Proof. Since f is balanced, we have $\hat{f}(\varnothing) = \mathbb{E}[f] = 0$. Therefore,

$$\begin{aligned}
\mathrm{Inf}(f) &= \sum_{S \subseteq [n]} |S| \hat{f}(S)^2 \\
&\geq \sum_{|S|=1} \hat{f}(S)^2 + 2 \sum_{|S| \geq 2} \hat{f}(S)^2 \\
&= 2 \sum_{|S|} \hat{f}(S)^2 - \sum_{|S|=1} \hat{f}(S)^2 \\
&= 2 - \sum_{|S|=1} \hat{f}(S)^2
\end{aligned}$$

□

Claim 9. *If $\hat{f}(i) \geq 1 - 2\epsilon$, then f is ϵ-close to χ_i; or if $\hat{f}(i) \leq -(1 - 2\epsilon)$, then f is ϵ-close to $-\chi_i$*

Proof.

$$\begin{aligned}
\hat{f}(i) &= \mathbb{E}[f(x)\chi_i(x)] \\
&= \Pr[f(x) = \chi_i(x)] - \Pr[f(x) \neq \chi_i(x)] \\
&= 1 - 2 \Pr[f(x) \neq \chi_i(x)]
\end{aligned}$$

□

Our proof replies on the observation that $\sum_i \mathrm{Inf}_i(f)(1 - \mathrm{Inf}_i(f))$ is very small:

Observation 10.

$$\begin{aligned}
\sum_i \mathrm{Inf}_i(f)(1 - \mathrm{Inf}_i(f)) &= \mathrm{Inf}(f) - \sum_i \mathrm{Inf}_i(f)^2 \\
&\leq \mathrm{Inf}(f) - \sum_i \hat{f}(i)^2 \\
&\leq 2\epsilon
\end{aligned}$$

From such an observation, we can guarantee there's some bit j that the influence of f on this bit is larger than $1 - 2\epsilon$:

$$\exists j : \mathrm{Inf}_j(f) \geq 1 - 2\epsilon$$

Otherwise, $\sum_i \mathrm{Inf}_i(f)(1 - \mathrm{Inf}_i(f)) > 2\epsilon \cdot \mathrm{Inf}(f) \geq 2\epsilon$.

Therefore, we would have

$$\begin{aligned}
\sum_{i \neq j} \hat{f}(i)^2 &\leq \sum_{i \neq j} \mathrm{Inf}_i(f) \\
&= \mathrm{Inf}(f) - \mathrm{Inf}_j(f) \\
&\leq 1 + \epsilon - (1 - 2\epsilon) \\
&\leq 3\epsilon
\end{aligned}$$

and

$$|\hat{f}(j)| \geq \hat{f}(j)^2 = \sum_i \hat{f}(i)^2 - \sum_{i \neq j} \hat{f}(i)^2 \geq 1 - \epsilon - 3\epsilon = 1 - 4\epsilon \tag{1}$$

□

Note that the proof above is relied on the condition that f is balanced. In the unbalanced case, let $\mu = \Pr_x[f(x) = -1] = 2^{-k}$, we know (by the edge-isoperimetric inequality) that the total influence of f is at least $2\mu \log(1/\mu) = k \cdot 2^{1-k}$. Also, the only functions that achieve this minimum total influence are indicator functions for subcubes. It was shown by David Ellis that if f has total influence close to this minimum, then it is also close to a subcube.[2] The FKN corollary establishes this result for $k = 1$.

3 From Structure to Influence

Now we consider the other direction: if the function is given in some format/structure, can we say something about its influence?

Specifically, suppose the boolean function is given in the disjunctive normal form:

Definition 11 (Disjunctive normal form)**.** *A formula ϕ is in disjunction normal form(DNF) if $\phi = T_1 \vee T_2 \vee ...T_s$ where the terms T are a conjunction of literals. The size of a DNF is the number of terms, denoted by s, and the width is the number of literals in the longest term, denoted by w.*

For example, $\phi_1 = (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_1 \wedge \bar{x}_4 \wedge x_5 \wedge \bar{x}_6)$ has size 2, width 4.

Just reminding you of the connection between DNF and the Boolean Hypercube (For a concrete definition of Boolean Hypercube, please turn to lecture 2): We can view the function f as a subset A of the hypercube, where A is the set of points on which $f(x) = 1$. Specifically, each term of a DNF could be taken as a set of points of the hypercube as well - the set T corresponding to a term is the set of inputs that satisfy the term. (For example, the set corresponding to the term $x_1 x_2 \bar{x}_3$ is the set of all inputs of the form $110x_4...x_n$.) Now the key observation is that those sets that correspond to terms have some structure: they must be a subcube of the hypercube: if a term contains t literals, then the subcube corresponding to this term has dimension $n - t$.

Indeed, if a DNF computes some function f, then the subcubes $T_1, T_2, ..., T_s$ that correspond to the terms of the DNF must cover all of A while not covering any of the points that are not in A. If we find a family of s subcubes that satisfy this condition, then f has DNF size complexity s; if we find a family of subcubes all of dimension at least $n - w$ that satisfy the condition, then f has DNF width complexity w.

Here comes the question: given a DNF with small width/size, can we get a bound of the influence?

Theorem 12 (Boppana [1])**.** *If a balanced function f is representable in DNF with width w, then $\mathrm{Inf}(f) \leq w$.* [1]

Proof.

$$\begin{aligned}
\mathrm{Inf}_i(f) &= \Pr_x[f(x) \neq f(x^{\oplus i})] \\
&= 2\Pr_x[(f(x) = -1) \cap (f(x^{\oplus i}) = 1)] \\
&= 2\Pr_x[f(x) = -1] \cdot \Pr_{x \in f^{-1}(-1)}[f(x^{\oplus i}) = 1] \\
&= \Pr_{x \in f^{-1}(-1)}[f(x^{\oplus i}) = 1]
\end{aligned}$$

[1] If f is unbalanced, the bound is $2w$.

Let $T(x)$ output the corresponding term of x in the DNF of f. [2] The observation is that, the chance of flipping the value of the output by flipping the ith bit of the input, is the same to the chance that the ith bit is shown in $T(x)$:

$$\begin{aligned}\mathrm{Inf}(f) &= \mathbb{E}_{x\in f^{-1}(-1)}[|\{i\in[n] : f(x^{\oplus i})=1\}|] \\ &\leq \mathbb{E}_{x\in f^{-1}(-1)}[|T(x)|]\end{aligned}$$

Since for all x, the length of $T(x)$ would not be longer than the width of the DNF, we have

$$\mathrm{Inf}(f)\leq w$$

□

After talking about the relationship between influence and the width of DNF, let's turn to the size of the DNF:

Theorem 13 (Boppana-Håstad)**.** *If f is balanced and represented in DNF with size s, then* $\mathrm{Inf}(f)\leq \log s+1$.

Notice that the original proof made use of Håstad Switching Lemma. By the random restriction method, if we want to bound to the total influence of a DNF of size s, it suffices to bound the total influence of DNFs of width $O(\log s)$.

Instead, today we are going to prove it in another way using Entropy Method: Recall the definition and some basic facts about entropy:

- $H(x)=\sum_i p(x_i)\log(p(x_i))$
- $H(x|y)=\sum_{i,j} p(x_i,y_j)\log(\frac{p(y_j)}{p(x_i,y_j)})$
- $H(x,y)=H(x)+H(y|x)$
- $H(x)\leq \log|\sup(p(x))|$

Proof. Let x be picked uniformly random from $f^{-1}(-1)$:

$$H(x,T(x))=H(x)+H(T(x)|x)$$

Notice that when x is picked, $T(x)$ is fixed, so $H(T(x)|x)=0$.

And $H(x)$ equals to the logarithm of the number of the terms satisfying $f^{-1}(-1)$, in the balanced case the number is $\frac{1}{2}\cdot 2^n$.

Therefore,

$$H(x,T(x))=H(x)=\log(\frac{1}{2}\cdot 2^n)=n-1 \tag{2}$$

Also,

$$\begin{aligned}H(x,T(x)) &= H(T(x))+H(x|T(x)) \\ &\leq \log(s)+\mathbb{E}_{y\in f^{-1}(-1)}[H(x)|T(y)]\end{aligned}$$

[2] For example, in the ϕ_1 mentioned above, if $x'=x_1x_2\bar{x}_3x_4x_5x_6$, then $T(x')=x_1\wedge x_2\wedge\bar{x}_3$

When we fix a term with length t in the DNF, the number of free bits corresponding to such a term is $n-t$. Therefore,

$$\begin{aligned} H(x, T(x)) &\leq \log(s) + n - \mathbb{E}_{y\in f^{-1}(-1)}[|T(y)| \\ &= \log(s) + n - \mathrm{Inf}(f) \end{aligned} \tag{3}$$

Combine (2) and (3) we got

$$n - 1 \leq \log(s) + n - \mathrm{Inf}(f)$$

□

References

[1] Ravi B. Boppana, The average sensitivity of bounded-depth circuits. Information Processing Letters, Volume 63, Issue 5, 15 September 1997, Pages 257-261

[2] David Ellis, Almost Isoperimetric Subsets of the Discrete Cube. Combinatorics, Probability and Computing, 20, 2011, Pages 363-380

[3] Ehud Friedgut, Gil Kalai, Assaf Naor, Boolean functions whose Fourier transform is concentrated on the first two levels. Advances in Applied Mathematics, Volume 29, Issue 3, October 2002, Pages 427-437