Numerical Methods

Lecture 2

CS 357 Fall 2013

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(Recall) Error

- Absolute error

$$|exact\ value\ - approximate\ value|$$

- Relative error

$$\frac{|exact\ value - approximate\ value|}{|exact\ value|}$$

Rounding and Chopping

- Round or Chop to 3 decimal places.

Number	Rounded	Chopped
0.2397	0.240	0.239
0.2375	0.238	0.237
0.2365	0.236	0.236
0.2344	0.234	0.234

Error Propagation

- Impact of arithmetic operations.

$$\tilde{a}_1 = a_1 + \delta_1, \tilde{a}_2 = a_2 + \delta_2$$

Addition:

Abs error $= (\tilde{a}_1 + \tilde{a}_2) - (a_1 + a_2) = (\delta_1 + \delta_2)$

Relative error* $= \frac{\delta_1 + \delta_2}{a_1 + a_2}$

* Relative error becomes infinite if $a_1 = -a_2$

Error Propagation

- Impact of arithmetic operations.

$$\tilde{a}_1 = a_1 + \ \delta_1, \tilde{a}_2 = a_2 + \delta_2$$

Multiplication:

$$\tilde{a}_1 \tilde{a}_2 = a_1 a_2 \left(1 + \frac{\delta_1}{a_1}\right)\left(1 + \frac{\delta_2}{a_2}\right)$$

$$= a_1 a_2 (1 + \rho_1)(1 + \rho_2)$$

Where $\rho_i = \frac{\delta_i}{a_i}$ is the relative error in $\tilde{a}_i, i = 1{,}2$

Error Propagation

- Impact of arithmetic operations.

$$\tilde{a}_1 = a_1 + \ \delta_1, \tilde{a}_2 = a_2 + \delta_2$$

Multiplication:

If $|\rho_i|$ is small such that $|\rho_1 \rho_2| \ll 1$ then

$$\tilde{a}_1 \tilde{a}_2 \cong a_1 a_2 (1 + \rho_1 + \rho_2)$$

And the relative error becomes

$$\rho_1 + \rho_2$$

Error Propagation

- Impact of arithmetic operations.

$$\tilde{a}_1 = a_1 + \delta_1, \tilde{a}_2 = a_2 + \delta_2$$

Division:

$$\frac{\tilde{a}_1}{\tilde{a}_2} = \frac{a_1}{a_2}\frac{(1+\rho_1)}{(1+\rho_2)}$$

$$= \frac{a_1}{a_2}\frac{(1+\rho_1)(1-\rho_2)}{1-{\rho_2}^2}$$

Where $\rho_i = \frac{\delta_i}{a_i}$ is the relative error in $\tilde{a}_i, i = 1,2$

Error Propagation

- Impact of arithmetic operations.

$$\tilde{a}_1 = a_1 + \ \delta_1, \tilde{a}_2 = a_2 + \delta_2$$

<u>Division</u>:

If $|\rho_i|$ is small such that $|\rho_1 \rho_2| \ll 1$ and ${\rho_2}^2 \ll 1$

$$\frac{\tilde{a}_1}{\tilde{a}_2} \cong \frac{a_1}{a_2}(1 + \rho_1 - \rho_2)$$

And the relative error becomes

$$\rho_1 - \rho_2$$

Error Propagation

Operation	Relative Error	Absolute Error
Addition	$\dfrac{\delta_1 + \delta_2}{a_1 + a_2}$	$(\delta_1 + \delta_2)$
Multiplication	$\rho_1 + \rho_2$	$a_1 a_2 (\rho_1 + \rho_2)$
Division	$\rho_1 - \rho_2$	$a_1 a_2 (\rho_1 - \rho_2)$

example

- Evaluate $y = \sqrt{x+\delta} - \sqrt{x}$
  - $x = 100$ and $\delta = 0.1$
  - using 2 decimals
- Solution

$$\sqrt{x+\delta} = \sqrt{100.1} = 10.0049987 \ldots$$

$$\tilde{y} = 10.00 \; - \; \sqrt{100} = 0.00^{*}$$

$$\left|\frac{\tilde{y}-y}{y}\right| = 1 \;\; \text{(catastrophic cancellation)}$$

*The subtraction is carried out exactly.

example

- Rewrite the formula

$$y = \left(\sqrt{x+\delta} - \sqrt{x}\right)\left(\frac{\sqrt{x+\delta} + \sqrt{x}}{\sqrt{x+\delta} + \sqrt{x}}\right)$$

$$= \frac{\delta}{\sqrt{x+\delta} + \sqrt{x}}$$

$$\tilde{y} = \frac{0.1}{10.0 + 10.0} = \frac{0.1}{20.0} = 0.005$$

$$\left|\frac{\tilde{y} - y}{y}\right| = 2.6 \times 10^{-4}$$

Simultaneous Equation Example

- Solve for y…

$$\begin{aligned} 0.1036x + \quad 0.2122y &= \quad 0.7381 \\ 0.2081x + \quad 0.4247y &= \quad 0.9327 \end{aligned}$$

- Carry only 3 significant digits of precision in the calculations.
- Repeat with 4 significant digits.

Simultaneous Equation Example

$$\begin{aligned} 0.1036x + \quad 0.2122y &= \quad 0.7381 \\ 0.2081x + \quad 0.4247y &= \quad 0.9327 \end{aligned}$$

Round to 3 significant digits

$$\begin{aligned} 0.104x + \quad 0.212y &= \quad 0.738 \\ 0.208x + \quad 0.425y &= \quad 0.933 \end{aligned}$$

Simultaneous Equation Example

Calculating the multiplier.

$$\alpha = \frac{0.208}{0.104} = 2.00$$

Calculating the y term multiplier in 2nd equ.

$$0.425 - (2.00)(0.212) = 0.001$$

Simultaneous Equation Example

$$
\begin{aligned}
0.104x + \quad 0.212y &= \quad 0.738 \\
0.208x + \quad 0.425y &= \quad 0.933
\end{aligned}
$$

Eliminate x-term in 2nd equation

$$
\begin{aligned}
0.104x + \quad 0.212y &= \quad 0.738 \\
0.0x + \quad 0.001y &= \quad -0.547
\end{aligned}
$$

$$\boxed{y = -547}$$

Simultaneous Equation Example

<u>Calculating the multiplier with four significant digits.</u>

Rounded value

$$\alpha = \frac{0.2081}{0.1036} \approx 2.009$$

<u>Calculating the y term multiplier in 2nd equ.</u>

$$0.4247 - (2.009)(0.2122) \approx 0.4247 - 0.4263 = -0.0016$$

Simultaneous Equation Example

$$0.1036x + 0.2122y = 0.7381$$
$$0.2081x + 0.4247y = 0.9327$$

Eliminate x-term in 2nd equation

$$0.1036x + 0.2122y = 0.7381$$
$$0.0x - 0.0016y = -0.5503$$

$$\boxed{y = 343.9}$$

Simultaneous Equation Example

$y = -547$

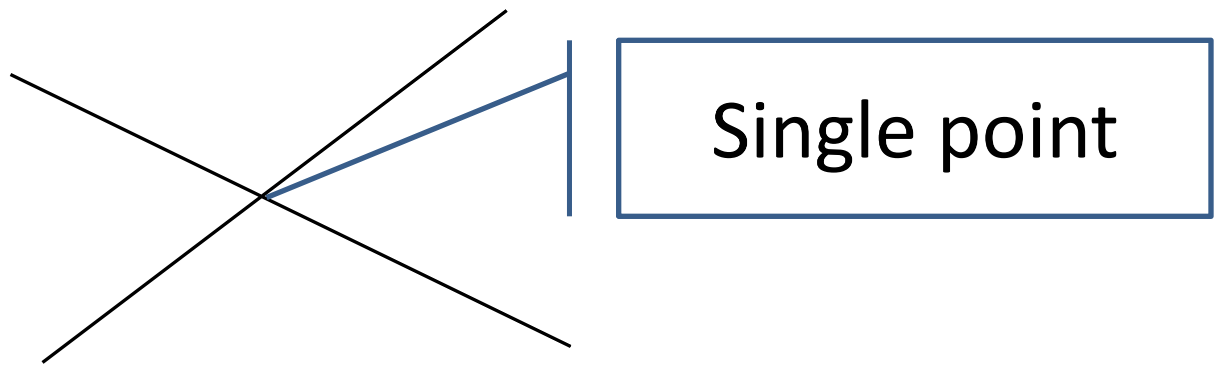
?

$y = 343.9$

Small change in coefficients caused a large change in solution.

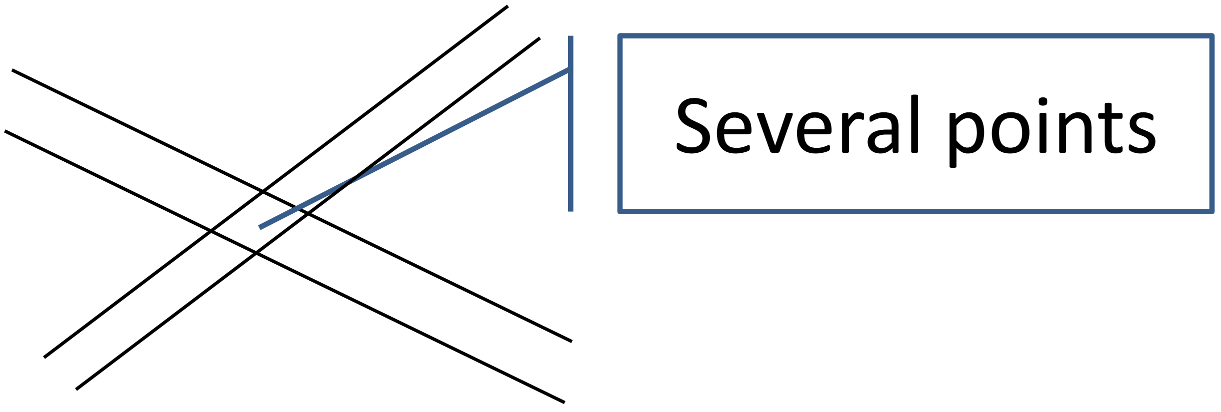
Infinite precision

2x2 linear system → Intersection of 2 lines in 2D.



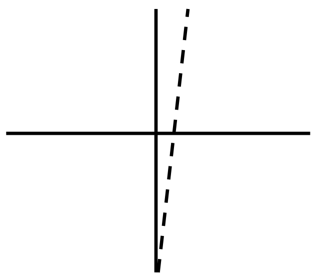
Finite precision

2x2 linear system → Intersection of 2 lines in 2D.

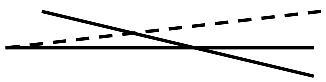


Ill conditioning

Well conditioned



Ill conditioned



Taylor Series

- Approximate a function $f(x)$ about a point c.
- We can write the Taylor Series of f at the point c:

$$f(x) \approx \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} f^k(c)$$

- Provided $f', f'', f''', \ldots$ exist at c.

Taylor Series for a polynomial

- Example

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$
$$c = 2$$

- Evaluate the derivatives of f at c and plug into the formula.

$$\begin{aligned} f(x) \approx\ & 207 + 396(x-2) + 295(x-2)^2 \\ & + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5 \end{aligned}$$

Horner's algorithm (aside)

- Evaluate:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

- Nested Multiplication:

$$p(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + x(a_n)) \ldots)$$

$$p(x) = \sum_{i=0}^{n} \left(a_i \prod_{j=1}^{i} x \right)$$

Taylor's theorem for f(x)

If the function f possesses continuous derivatives of orders 0,1,2,…,(n+1) in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(c)}{k!}(x - c)^{k} + E_{n+1}$$

Where the error term can be given in the form

$$E_{n+1} = \frac{f^{(n+1)}(\varepsilon)}{(n + 1)!}(x - c)^{n+1}$$

Taylor series approximation

Approximate $f(x) = \frac{1}{(1-x)}$.

$$f(x) = f(c) + (x-c)f'(c) + (x-c)^2 f''(c) + \ \ldots$$

With $c = 0$

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \cdots$$

2nd order approximation: $\frac{1}{(1-x)} \approx 1 + x + x^2$

Taylor series approximation

How many terms are required for error to be less than $2.0\ x\ 10^{-8}$ at $x = 1/2$?

$$E_{x=1/2} = \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)^{n+1}}{1 - 1/2}$$

$$= 2 \cdot \left(\frac{1}{2}\right)^{n+1} < 2 \times 10^{-8}$$

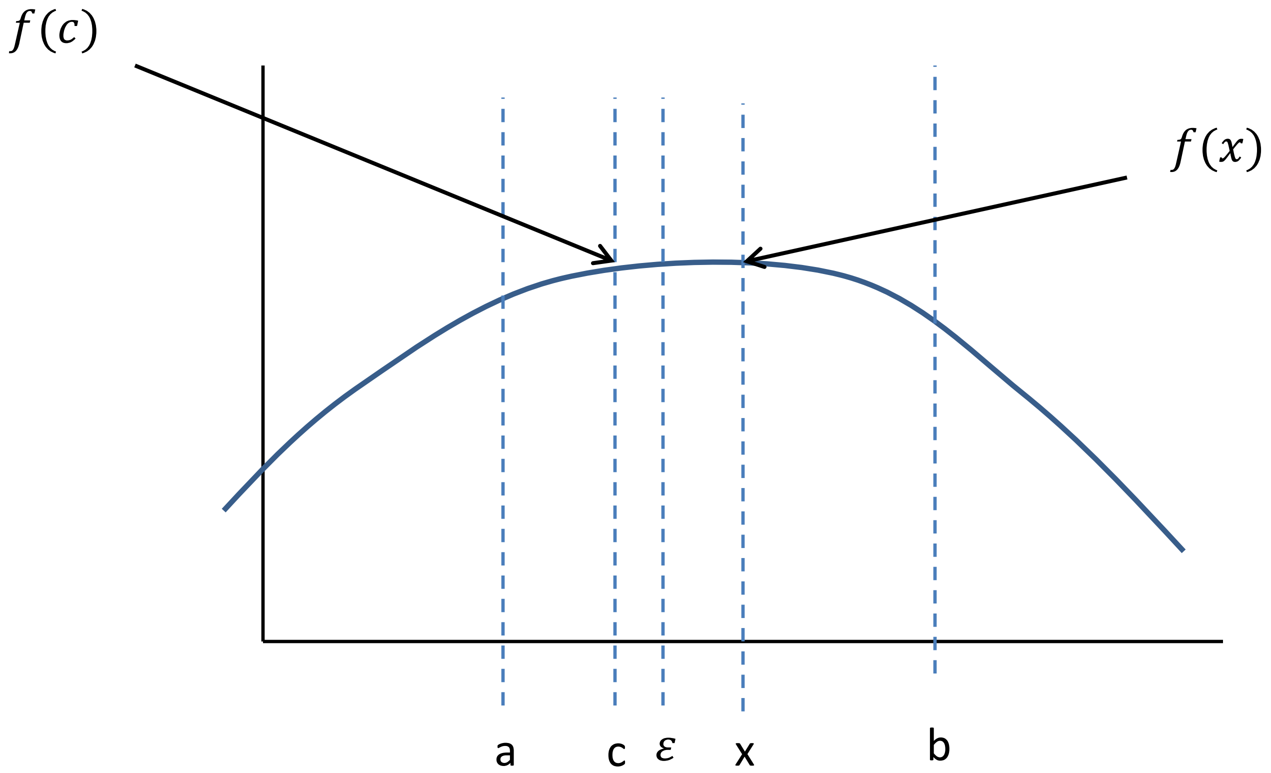
Taylor series approximation

$$2 \cdot \left(\frac{1}{2}\right)^{n+1} < 2 \times 10^{-8}$$

$$(n+1)\log_{10}\left(\frac{1}{2}\right) < -8$$

$$(n+1) > \frac{-8}{\log_{10} 1/2} \approx 26.6 \ \rightarrow n > 26$$

Taylor's theorem for f(x)



Mean-Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and possesses a derivative at each point on the open interval (a, b), then

$$f(b) = f(a) + (b - a)f'(\varepsilon)$$

for some ε in (a, b).

Taylor theorem for f(x+h)

If the function f possesses continuous derivatives of orders 0 through $n+1$ in $[a, b]$, then for any x in $[a, b]$,

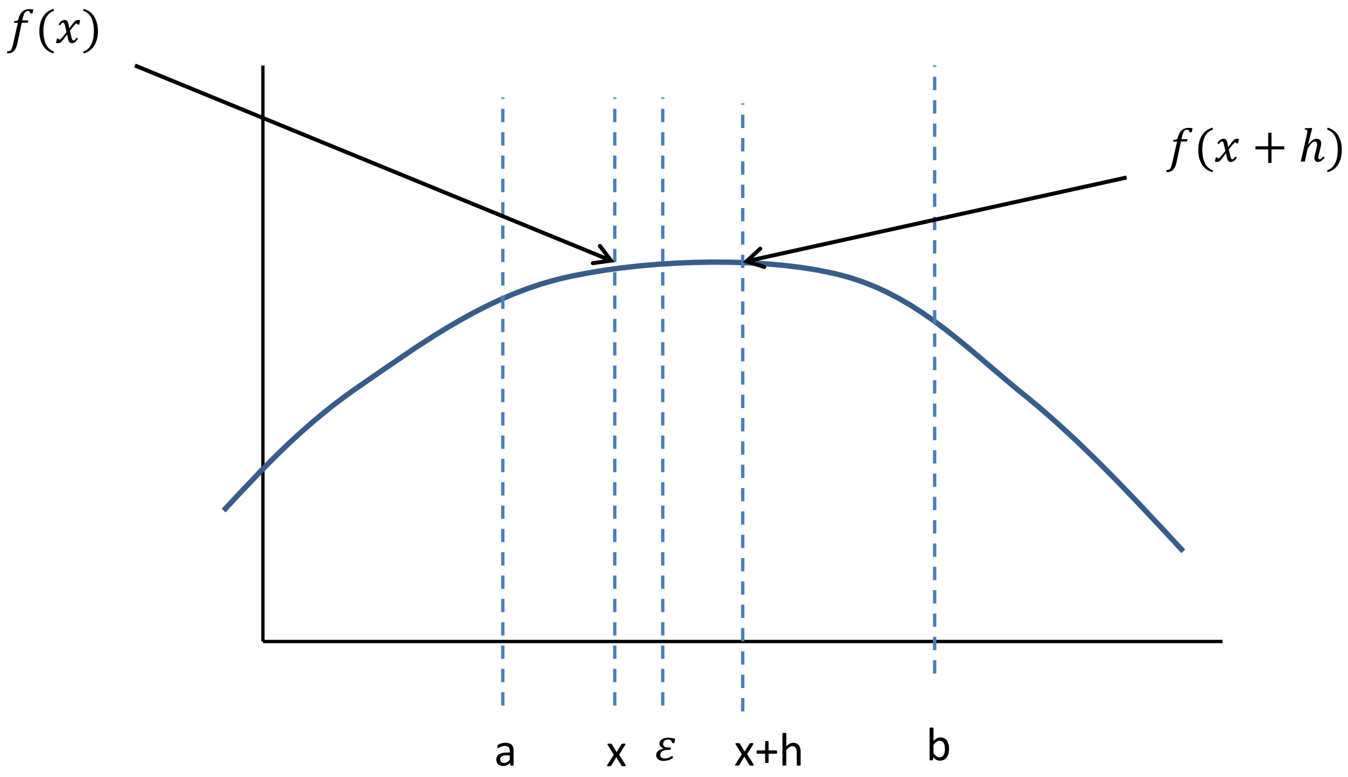
$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

Where h is any value such that x+h is in I and where

$$E_{n+1} = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{n+1}$$

For some ε between x and $\mathrm{x+h}$.

Taylor's theorem for f(x+h)



Big O

- The error term depends on h explicitly and because ε depends on h.
- $E_{n+1} \to 0$ as $h^{n+1} \to 0$.
- For large n this convergence is rapid.

$$E_{n+1} = \vartheta(h^{n+1})$$

Taylor series for f(x+h)

Expand $\sqrt{1+h}$ in powers of h.

Evaluate $\sqrt{1.00001}$, $\sqrt{0.99999}$

$$f(x) = \sqrt{x}, with\ x = 1$$

To 2nd order:

$$\sqrt{1+h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3\varepsilon^{-5/2}$$

$$1 < \varepsilon < 1 + h$$

Taylor Series for f(x+h)

$$h = 10^{-5} \to \sqrt{1.00001} \approx 1 + 0.5 \times 10^{-5} - 0.125 \times 10^{-10}$$
$$= 1.00000\ 49999\ 87500$$

Similarly

$$h = -10^{-5} \to \sqrt{0.99999} \approx 1 - 0.5 \times 10^{-5} - 0.125 \times 10^{-10}$$
$$= 0.99999\ 49999\ 8750$$

Since $1 < \varepsilon < 1 + h$ the absolute error does not exceed

$$\frac{1}{16} h^3 \varepsilon^{-5/2} < \frac{1}{16} 10^{-15}$$

The 2nd order approximation retains full 15 decimals of precision.

Alternating Series Theorem

If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series:

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

Converges; that is,

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty} S_n = S$$