Elimination Matrices

Elimination matrices are a matrix operator that will zero entries in a column vector. It is important to understand that Elimination matrices are never formed explicitly or used directly in numerical calculations. The matrix is a tool for understanding Gaussian elimination and other matrix factorization methods. Consider the 3 X 3 matrix below and its factorization.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 7 & 9 \\ 1 & 6 & 3 \end{bmatrix}$$

The first stage of Gaussian elimination requires us to use multiples of the first element in the first row (the pivot element) to zero out the first element in the 2nd and 3rd rows respectively. Performing this operation gives us:

$$M_1 A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 7 \\ 0 & 5 & {}^{5}/_{2} \end{bmatrix}$$

Here we have multiplied each element in the first row of A by -2 and added it to the second row of A. We then multiply the first row of A by -1/2 and add it to the third row of A. M_1 is the first elimination matrix and it is:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -{}^{1}/_{2} & 0 & 1 \end{bmatrix}$$

The nonzero elements below the diagonal of $M_1$are the multipliers used to eliminate the respective rows of A. For example the multiplier used to eliminate the first element in the second row of A is -2. Similarly the multiplier used to eliminate the first element of the third row of A is -1/2. One can verify the validity of this matrix by multiplying the original matrix A by it.

We construct the next elimination matrix by adding a multiple of the second row of $M_1 A$ to the third row in order to zero the 5 in the third row. The multiplier is -5/3 and we after we perform the stage in the elimination we have:

$$M_2 M_1 A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -{}^{55}/_{6} \end{bmatrix}$$

Now $M_2$is the second elimination matrix and has this form:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/3 & 1 \end{bmatrix}$$

Notice that multiplying a matrix by an elimination matrix M_k only changes elements in rows of A greater than k. For example, M_1 only changes elements in rows $k > 1$. Furthermore each row of the elimination matrix forms a linear combination of two rows of the matrix it multiplies. For example the third row of M_1 forms a linear combination of rows 1 and 3 of A. It adds a multiple of row 1 to row 3.

In order to undo this operation one would have to subtract the same multiple of row 1 to the resultant row 3. This idea leads to the inverse of an elimination matrix being the matrix itself with the sign of the multipliers reversed. Consider the inverse of the first elimination matrix:

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

So in the end we have, for our example:

$$M_2 M_1 A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -55/6 \end{bmatrix}$$

And

$$A = M_1^{-1} M_2^{-1} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -55/6 \end{bmatrix}$$

Where it can be show by matrix multiplication that

$$M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 5/3 & 1 \end{bmatrix}.$$

Notice that this matrix is unit lower triangular and the columns are the union of the columns of the product matrices. Denoting this matrix by L and we can write:

$$A = M_1^{-1} M_2^{-1} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -55/6 \end{bmatrix} = LU$$

Where U is upper triangular. This is the expression we were after in the first place.