Lecture 27

Final Exam Review

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CS357 Final Exam

Content:

- 50 multiple choice questions
  - 20 questions from lectures 20 - 26. (not including conjugate gradient)
  - 15 questions from Exam1
  - 15 questions from Exam2

Logistics:

- December 20, 7:00 pm to 10:00 pm (Please arrive early so as to have the entire 3 hours)
- Last Name beginning with A - L $\rightarrow$ 100 MSEB.
- Last Name beginning with M - Z $\rightarrow$ 1404 Siebel Center

Rules:

- One empty seat between each student.
- All electronic devices turned off and stowed.
- One 8.5 x 11 crib sheet allowed as per mid term exams.
- Photo ID required.
- All other standard University final exam rules apply. (no cheating etc.)

Exam 1 questions

(True/False) If $\epsilon_{mach} = 10^{-9}$ and $k(A) = 10^{7}$, then we can guarantee the accuracy of the following solution

$$\hat{x} = \begin{bmatrix} 3.01 \\ 2.12 \\ 5.81 \end{bmatrix}$$

1. True
2. False

Solution: B

Exam 1 Questions

- If A and b are stored to machine precision ε_m, the numerical solution to $Ax = b$ by any (good) variant of Gaussian elimination is correct to d decimal digits where

$$d = |\log_{10}(\varepsilon_m)| - \log_{10}(\kappa(A))$$

Exam 1 Questions

$$d = |\log_{10}(\varepsilon_m)| - \log_{10}\left(\kappa(A)\right)$$

Exam Problem:
$\varepsilon_m = 10^{-9}$. $\kappa(A) = 10^7$ the elements of the solution vector will have

$$\begin{aligned} d &= |\log_{10}(10^{-9})| - \log_{10}\left(10^7\right) \\ &= 9 - 7 \\ &= 2 \end{aligned}$$

correct digits

Exam 1 Questions

(True/False) The gap between 2 and the next larger number is equal to the gap between 3 and the next larger number in IEEE double-precision.

1. True
2. False

Solution: A

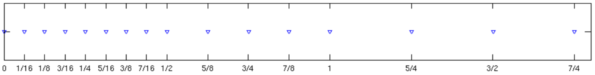
Exam 1 Questions

- Spacing between adjacent floating point numbers with the same exponent is the same.
- Representation of 2.0 and 3.0 in IEEE double precision has the same exponent part.
- The distance between 2.0 and the next larger number is the same as the distance between 3.0 and the next larger number.

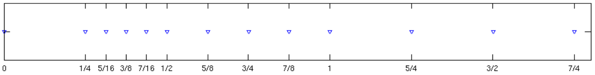
Exam 1 Questions

Consider the binary number

$$x = \pm(0.b_1 b_2 b_3) \times 2^{\pm k}$$



Normalized $b_1 = 1$



Exam 2 Questions

Which of the following methods is Gauss-Seidel most closely related to?

1. Jacobi
2. SOR
3. Gaussian Elimination
4. Conjugate Gradient

The answer is 2 SOR.

Exam 2 Questions

Consider the Gauss-Seidel method. If we construct the next iterate for x in the following way we have SOR method.

$$x^{(k)} = (D - \omega L)^{-1} \left[\omega U + (1-\omega)D\right] x^{(k-1)} + (D - \omega L)^{-1} \omega b$$

Component-wise:

$$x_i^{(k)} = \omega \left[- \sum_{j=1, j<i}^{n} \left(\frac{a_{ij}}{a_{ii}}\right) x_j^{(k)} - \sum_{j=1, j>i}^{n} \left(\frac{a_{ij}}{a_{ii}}\right) x_j^{(k-1)} + \frac{b_i}{a_{ii}} \right] + (1-\omega) x_i^{k-1}$$

Exam 2 Questions

Determine the weights for the quadrature formula when the interval is [-1,1] and the nodes are -1, 0 and 1.

1. $A_0 = \frac{1}{3}, A_1 = \frac{4}{3}, A_2 = \frac{1}{3}$
2. $A_0 = \frac{2}{3}, A_1 = \frac{-4}{3}, A_2 = \frac{2}{3}$
3. $A_0 = \frac{1}{3}, A_1 = \frac{-4}{3}, A_2 = \frac{1}{3}$

The correct answer is 1.

Exam 2 Questions

Use the method of undetermined coefficients to obtain the weights. We know A_0, A_1, A_2 satisfy

$$\int_{-1}^{1} f(x)\,dx = A_0 f(-1) + A_1 f(0) + A_2 f(1)$$

exactly when $f(x) = ax^2 + bx + c$. The expression will be exact for all degree 2 polynomials if it is exact for 1, x, and x^2.

$$\int_{-1}^{1} dx = 2 \rightarrow A_0 + A_1 + A_2 = 2$$

$$\int_{-1}^{1} x dx = 0 \rightarrow A_2 - A_0 = 0$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} \rightarrow A_0 + A_2 = \frac{2}{3}$$

Exam 2 Questions

Which of the following approximations of $f'(x)$ has the error term $\frac{-2}{3}h^2 f'''(\varepsilon)$?

1. $\frac{1}{4h}[f(x+2h) - f(x-2h)]$
2. $\frac{1}{4h}[f(x+3h) - f(x-h)]$
3. $\frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$
4. $\frac{1}{2h}[f(x+2h) - f(x-h)]$

The correct answer is 1.

Exam 2 questions

Expand in Taylor series:

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{iv}(x) + \cdots$$

$$f(x+2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{iv}(x) - \cdots$$

Solve for $f'(x)$

$$f'(x) = \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] - \frac{2h^2}{3} f'''(\xi)$$

Exam 2 Questions

Which of the following is a necessary and sufficient condition for the convergence of an iterative method for solving the system $Ax = b$.

1. Diagonal dominance of the iteration matrix.
2. The eigenvalues of the iteration matrix lie within the open unit disk.
3. Diagonal dominance of A.
4. All the eigenvalues of A are real.

The correct answer is 2.

Exam 2 Questions

Look again at the iteration

$$x^{(k)} = x^{(k-1)} + Q^{-1} r^{(k-1)}$$

Looking at the error:

$$x - x^{(k)} = x - x^{(k-1)} - Q^{-1} r^{(k-1)}$$

Gives

$$e^{(k)} = e^{(k-1)} - Q^{-1} A e^{(k-1)}$$

or

$$e^{(k)} = (I - Q^{-1} A) e^{(k-1)}$$

or

$$e^{(k)} = (I - Q^{-1} A)^k e^{(0)}$$

Exam 2 Questions

We want

$$e^{(k)} = (I - Q^{-1}A)^k e^{(0)}$$

to converge.

When does $a_k = c^k$ converge? .....when $|c| < 1$

Likewise, our iteration converges

$$\begin{aligned} \|e^{(k)}\| &= \|(I - Q^{-1}A)^k e^{(0)}\| \\ &\leqslant \|I - Q^{-1}A\|^k \|e^{(0)}\| \end{aligned}$$

when $\|I - Q^{-1}A\| < 1$.

Markov Chains

- Transition matrix
- Probability vector
- Definition of:
  - Markov Matrix
  - Markov Chain
  - Absorbing state
- Relationship between eigenvalues and Markov chains.

Eigenvalues

- Power method
- Power method with normalization
- Inverse Power method.
- Finding largest and smallest eigenvalues

Singular Value Decomposition

- Singular Value Decomposition
- Similar matrices
- Relationship between eigenvalues of $A^T A$ and singular values of A.
- What is SVD used for.

Least Squares

- Overdetermined systems
- Normal equations
- Data fitting
- Condition of the normal equations
- Using QR and SVD to solve linear least squares problems
- Orthogonal Martices
- Gram-Schmidt Orthogonalization
- Orthogonal projections

Monte Carlo

- Expected value of a variable
- Variance
- Estimating expected value by experiment (rolling dice for example)
- Computer model of experiments with random numbers.
- Uses of Monte Carlo:
  - computing volumes
  - simulation (rolling dice, birthday problem)

Randomness

- Pseudorandom numbers
- Random number generators
- Linear Congruential Generators
- Periodicity
- Sensitivity to coefficients

FFT

- Least squares with trigonometric polynomials
- Fourier Coefficients
- Continuous and Discrete inner products
- Properties of inner products
- Orthogonal functions
- Orthonormality
- Be able to calculate the fourier coefficients given f and n.
- Roots of Unity
- Perform discrete fourier transform (DFT)
- Identify DC and Nyquist components.
- Operation count for DFT and FFT