CS357:Final Exam Study Guide

Use these questions as a guide. The final exam will cover all material. The exam is Friday, December 20. 7 pm - 10 pm. Bring your ID. You are allowed a single crib sheet as per the midterm exams.

The questions do not represent all questions on the final exam.

1. Using a Taylor Series approximation of $f(x) = \sin(x)$ about $x_0 = 0$, how many **nonzero** terms are necessary in order to guarantee an absolute error of 10^{-7} at $x = 1/10$? You may find the following useful:

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800

   (a) 2 terms
   (b) 4 terms
   (c) 6 terms
   (d) 8 terms

2. Which one of the following is true about the IEEE-754 floating point standard?
   (a) There are only a finite number of machine representations.
   (b) Adding a list of numbers does not depend on the order in which they are summed.
   (c) The floating point representations are uniformly distributed on the real line with a hole at zero.
   (d) The methods of rounding (e.g. chopping, round-to-nearest, etc) result in the same relative error.

3. Consider the following Python script:

```

x=2**(-1022);

while (x+1.0 <= 1.0):
    x = 2*x
end

print x
```

   What is the outcome?
   (a) The lowest representable floating point is returned since the while loop is not entered.
   (b) $+\infty$ is returned after the while loop overflows.
   (c) ϵ_m is returned
   (d) the while loop never returns

4. **True or False** Machine epsilon, ϵ_m, is the smallest representable positive floating point number in double precision.

5. **True or False** The bisection method converges for finding a root of $f(x) = 2x^5 - 1$ in the interval $[0, 1]$.

6. **True or False** Suppose that bisection method converges to a root of a function $g(x)$ in the interval $[0, 1]$. How many iterations are needed to ensure that the approximate root is accurate to within 10^{-6}? ($\log_{10}(2) \approx .3$).
   (a) 5
   (b) 10
   (c) 15
   (d) 20
   (e) 25

7. Given a starting guess of $x_0 = 5$, what is an approximation to a root of $f(x) = x^2 - 4$ using one step of Newton's method?

   (a) 2
   (b) 2.1
   (c) 2.9
   (d) 95/21
   (e) 7.1

8. Given starting guesses of $x_0 = 0$ and $x_1 = 3$, what is an approximation to a root of $f(x) = x^2 - 4$ using one step of the Secant method?

   (a) -5/3
   (b) 2
   (c) 4/3
   (d) 10/3
   (e) $+\infty$

9. **True or False** When Newton's method converges it is always quadratic convergence.

10. **True or False** The convergence rate of the Secant method is slower than Newton, but requires fewer function evaluations per iteration.

11. When interpolating a distinct data set of 6 values, the degree of a Lagrange basis function used in interpolation is

   (a) 2
   (b) 3
   (c) 4
   (d) 5
   (e) 6

12. The Newton form of the quadratic interpolant of $f(x) = \frac{12}{2+x}$ using $x = 0, 1, 2$ is

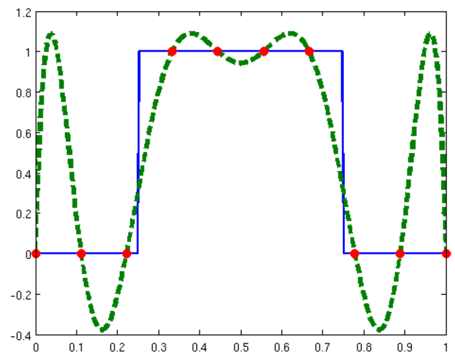
   (a) $p_2(x) = 6 - 2x + 1/2x(x-1)$
   (b) $p_2(x) = 6 - 4x + 3x(x-1)$
   (c) $p_2(x) = 6 - 2x + 3/2x(x-1)$
   (d) $p_2(x) = 6x(x-1) - 2x + 1/2$

13. **True or False** The Newton form of a polynomial interpolant of a given set of data is *less expensive* to construct *and* is better conditioned on evaluation than the monomial form of the polynomial.

14. Which one of the following is **not** true of the Lagrange form of an interpolating polynomial?

   (a) The construction of the coefficients is inexpensive.
   (b) The evaluation at any point is efficient through Horner's method.
   (c) The evaluation at any point is well conditioned.

15. Consider the following interpolant (dashed) of the step function (solid) through the uniform nodes (circles).



Which one of the following is **not** likely?

(a) Increasing the number of points (with uniform spacing) will improve the dashed interpolant.

(b) The interpolant can be improved through non-uniform nodes (e.g. Chebychev)

(c) A spline will resolve the step function more accurately.

16. What is the central difference approximation to $f'(x)$ at $x = 2$ using a stepsize of $h = 1$ if $f(x) = x^3$?

(a) 7

(b) 13

(c) 19

(d) 35

17. Suppose we use numerical differentiation of $f(x) = \cos(x)$ at $x = \pi/6$ and find the following results:

h	$D_h f(x)$	Error
0.1	-0.54243	0.04243
0.05	-0.52144	0.02144
0.025	-0.51077	0.01077
0.0125	-0.50540	0.00540
0.00625	-0.50270	0.00270
0.003125	-0.50135	0.00135

The convergence is

(a) $\mathcal{O}(h^{1/2})$

(b) $\mathcal{O}(h)$

(c) $\mathcal{O}(h^2)$

(d) $\mathcal{O}(h^3)$

18. **True or False** The degree of precision of the basic Trapezoid method is 1 while the degree of precision of the basic Simpson method is 2.

19. **True or False** The 4-point Gaussian Quadrature for the integral

$$\int_{-1}^{1} 3x^6 + 4x^5 + x^4 + x - 1\, dx$$

is exact.

20. What is the numerical approximation to

$$\int_{-1}^{1} \frac{3}{2+x}\, dx$$

using basic trapezoid rule?

(a) 0
(b) 2
(c) 4/3
(d) 4

21. Given k knots and data points, what is the cost of computing the natural cubic spline?

    (a) $\mathcal{O}(k)$
    (b) $\mathcal{O}(k^2)$
    (c) $\mathcal{O}(k^3)$
    (d) $\mathcal{O}(1)$

22. **True or False** A cubic spline for k knots and data points is unique.

23. Suppose that A is a dense $n \times n$ matrix and that v is an $n \times 1$ vector. What is the computational cost of computing Av?

    (a) $\mathcal{O}(1)$
    (b) $\mathcal{O}(n)$
    (c) $\mathcal{O}(n^2)$
    (d) $\mathcal{O}(n^3)$

24. Suppose that A is a sparse (CSR) $n \times n$ matrix with at most 4 entries per row and that v is an $n \times 1$ vector. What is the computational cost of computing Av?

    (a) $\mathcal{O}(1)$
    (b) $\mathcal{O}(n)$
    (c) $\mathcal{O}(n^2)$
    (d) $\mathcal{O}(n^3)$

25. **True or False** The following matrix ill-conditioned

$$A = \begin{bmatrix} 4.4e10^{-8} & 0 \\ 0 & 1.0e10^{7} \end{bmatrix}$$

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$$A = \begin{bmatrix} 4.4e10^{8} & 0 \\ 0 & 1.0e10^{7} \end{bmatrix}$$

27. **True or False** Well conditioned matrices do **not** benefit from partial pivoting.

28. What is the added cost of scaled partial pivoting over naive Gaussian elimination?

    (a) $\mathcal{O}(1)$
    (b) $\mathcal{O}(n)$
    (c) $\mathcal{O}(n^2)$
    (d) $\mathcal{O}(n^3)$

29. Using **scaled partial pivoting** to solve the following matrix problem, what is the first pivot element?

$$Ax = \begin{bmatrix} 50 & 2 & 3 & 100 \\ 1 & 0 & 0 & 1000 \\ 2 & 1 & 1 & 10^{-2} \\ 10^{-2} & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = b$$

(a) Row 1, Column 1
(b) Row 1, Column 4
(c) Row 2, Column 1
(d) Row 3, Column 1
(e) Row 4, Column 1

30. Suppose that we have already computed the LU factorization of a matrix A. To solve $Ax = b$, first solve $Ly = b$ for y followed by solving $Ux = y$ for x. What is the cost of this process (given the LU factorization beforehand)?

(a) $\mathcal{O}(1)$
(b) $\mathcal{O}(n)$
(c) $\mathcal{O}(n^2)$
(d) $\mathcal{O}(n^3)$

31. Consider the iterative solution to $Ax = b$ using the Jacobi method with

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ -3 \\ -7 \\ 9 \end{bmatrix} \quad x_{exact} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

The method will converge for any initial guess

(a) True
(b) False

32. **True or False** The Power Method will converge for the following matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

33. **True or False** The convergence of Monte Carlo integration is the same in any dimension.

34. If A is an $m \times n$ matrix, what is the maximum number of nonzero singular values it can have?

(a) $\min\{m, n\}$
(b) $\max\{m, n\}$
(c) n
(d) m

35. Suppose we compute the SVD of a 2×3 matrix $A = USV^T$. Which one of the following is a candidate for U?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$
(d) All of the above.
(e) None of the above.

36. Apply one iteration of the normalized power iteration to the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

using $x_0 = [1\ 1]^T$ as an initial guess. After normalization, one entry of the resulting vector x_1 will be 1. What is the value of the other entry?

(a) 0.0
(b) 0.2
(c) 0.4
(d) 0.6
(e) 0.8

37. **True or False** A symmetric matrix will always have positive, real eigenvalues.

Questions to ask yourself:

38. Define *round-off* error and *truncation* error. What is the difference?

39. What is *machine epsilon*?

40. How many terms of a Taylor series expansion (about $c = 0$) should we use to ensure that $f(x) = e^x$ is approximated to single precision accuracy ($\approx 10^{-7}$) when evaluating at the point $x = \frac{1}{10}$?

41. Interpolate $\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 0 & 0 & 4 & 24 \end{array}$ using Newton interpolation. Detail two benefits of Newton-based interpolation (versus using monomials and Lagrange basis functions).

42. Consider the integral $\int_0^1 1 + x^2\, dx$. Graphically depict numerical integration using basic Trapezoid and 2-point Gauss Quadrature (2 different graphs). Is one of the these methods more accurate? Is either of them *optimal* in some sense? Be sure to incorporate *degree-of-precision* into your discussion.

43. The following table results from numerically approximating a mathematical operation based on some mesh spacing h (here the nodes are evenly spaced $x_0, \ldots, x_n$ in $[a, b]$ with $h = x_1 - x_0$). What do the results tell us about accuracy of the method? (for example order).

h	approximation value	Error
0.1	-0.49916708	-0.0008329
0.05	-0.49979169	-0.0002083
0.025	-0.49994792	-0.00005208
0.0125	-0.49998698	-0.00001302
0.00625	-0.49999674	-0.000003255

44. Give an example of a matrix that benefits from partial pivoting (either regular partial pivoting or scaled partial pivoting). Explain. Give an example of a matrix with both small and large entries that is *well conditioned* and one that is *ill-conditioned*. Justify.

45. Suppose we wish to integrate $\int_5^{10} e^{(x^2/100)}\, dx$ using Monte Carlo techniques. Sketch an algorithm and discuss the convergence.

46. Will Jacobi iteration converge for the following matrix problem?

$$\begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & 3 & -1 & \\ & & \ddots & & \\ & & \vdots & & \\ & & & -1 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

47. Run one step of Gauss-Seidel and one step of Jacobi iteration for the following problem $Ax = b$ with $x_0 = [1, 1, 1]^T$, $b = 0$, and

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

48. What conditions on A are necessary for a Conjugate Gradient method?

49. Is the following matrix acceptible?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

50. True or False, the convergence speed of Jacobi and Gauss-Seidel iteration depends on the condition number of the matrix A. (for Conjugate Gradient?)

51. True or False, the power method finds the smallest eigenvalue-eigenvector pair of the matrix A.

52. Run a few steps of power iteration on the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with initial guess $[1, 1, 1]^T$. Then run again with $[-1, -1, -1]^T$. What is happening? What conditions do we need on A for the power method to converge?

53. True or False, the inverse power method find the smallest eigenvalue-eigenvector pair of the matrix A.

54. Consider the SVD of matrix A which is $n \times m$. What are the dimensions of each matrix in $A = USV^T$?

55. What property does U have? The columns of U are the eigenvectors of?

56. What property does V have? The columns of V are the eigenvectors of?

57. What is the SVD of $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$?

58. Suppose all but 3 singular values of an $n \times n$ matrix are less than 10^{-15}. What is the cost saving of storing a matrix in SVD form?

59. Suppose $Ax = b$ is an $n \times m$ overdetermined system. The least-squares solution finds what?

60. Compare the condition number of the A versus $A^T A$.

61. Should we solve $A^T Ax = A^T b$ directly? What else should we use?

62. Let $A = QR$. What are the properties of Q and R? How is Q constructed?

63. Suppose $y = [0, 1, 0, 0, \ldots, 0]$. What is the inverse FFT of y?

64. What is the general cost of an FFT?

65. What is the cost of an FFT if the length of the sequence is $n = 2^k$?

66. What is the first component of $DFT(x)$?