Math 55 Worksheet

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Adapted from worksheets by Rob Bayer, Summer 2009 and Noah Forman Summer 2011

Predicates and Quantifiers

1. If $P(x)$ is the statement "x > 0" and $G(x, y)$ is the statement $x^2 \geq y$, determine the truth value of each of the following

   (a) $P(1)$ (b) $G(2, -3)$ (c) $P(-1) \to G(-2, 1)$

2. Using the same predicates as above, determine the truth values of each of the following statements if the domain is the set of all real numbers.

   (a) $\forall x P(x)$ (b) $(\exists x P(x)) \land (\forall x G(x, 0))$ (c) $\exists y G(2, y)$

3. Suppose the domain of $P(x)$ consists of the integers 0,1,2,3. Re-write each of the following statements without using quantifiers:

   (a) $\forall x P(x)$ (b) $\forall x\,(x \neq 3 \to \neg P(x))$

4. Let $S(x)$ be the statement "x is a student," $L(x)$ be "x lives in Japan" $J(x)$ be "x speaks Japanese." Translate each of the following into English or into logic symbols as appropriate. The domain is the set of all people.

   (a) $\exists x\,(L(x) \land S(x))$.
   (b) $\forall x((L(x) \land \neg S(x)) \to J(x))$.
   (c) $\exists x(S(x) \land L(x) \land \neg J(x))$.
   (d) There is a Japanese speaking student.
   (e) Not all speakers of Japanese live in Japan.
   (f) Some students live in Japan, but some don't.

5. Determine whether each of the following pairs of sentences are equivalent. If so, explain why. If not, give an example of predicates and domains where they differ.

   (a) $\exists x(P(x) \land Q(x))$; $\exists x P(x) \land \exists x Q(x)$.
   (b) $\forall x(P(x) \land Q(x))$; $\forall x P(x) \land \forall x Q(x)$.
   (c) $\exists x(P(x) \to Q(x))$; $\exists x P(x) \to \exists x Q(x)$.

Nested Quantifiers and Restricted Domains

1. Let $T(x, y)$ be "x is taking y", $L(x, y)$ be "x likes y", $R(x, y)$ be "x is required to take y." If the domain for x is all students and they domain for y is all classes, translate each of the following between English and Logic:

   (a) $\forall x \exists y L(x, y)$.
   (b) $\exists y \forall x \neg L(x, y)$.
   (c) $\forall y \exists x R(x, y)$.
   (d) $\exists x \exists y(T(x, y) \land \neg L(x, y))$.
   (e) Every student is required to take at least one class.
   (f) Some student likes all of his/her current classes.
   (g) Some student only likes courses which he/she isn't required to take.

2. Determne the truth value of each of the following statements. The domain is the set of all real numbers

   (a) $\forall x \exists y(x > y)$.
   (b) $\exists x \exists y(x \geq y \land y \geq x)$.
   (c) $\forall x \exists y(x = y^2)$.
   (d) $\forall x \forall y \exists z(x > y \to x > z > y)$.
   (e) $\exists x \exists y(x + y = 1 \land x - y = 3)$.
   (f) $\forall \epsilon > 0\ \exists \delta > 0\ \forall x\ (|x - 3| < \delta \to |x^2 - 9| < \epsilon)$.

3. Re-write each of the following so that negation symbols appear only after predicates.

   (a) $\neg \exists x \forall y P(x, y)$.
   (b) $\neg \exists x \neg P(x)$.
   (c) $\neg \forall x \forall y(P(x, y) \to \neg Q(x, y))$.
   (d) $\neg((\forall x P(x)) \lor \exists y \forall z(R(y, z) \land S(y, z)))$.

4. ★ The symbol $\exists! x P(x)$ stands for "there exists one and only one x such that P(x) is true" and is often pronounced "there is a unique x..." Show that $\exists! x P(x)$ can be rewritten using just regular quantifiers.

Puzzles

1. Consider the following set of four statements:

   (a) One of these statements is false

   (b) Two of these are false

   (c) Three of these are false

   (d) Four of these are false

   Which of the above, if any, are true?

2. Two bicyclists enter opposite ends of a 100-foot long tunnel that is only wide enough for one bike. One is travelling 10 ft/s and the other is travelling 5 ft/s in the opposite direction. A bird flying 20 ft/s enters the tunnel just in front of the 10 ft/s cyclist. When the bird gets to the other cyclist, it immediately turns around and flies back towards the 10 ft/s cyclist. When the bird gets back to him, it turns around again, etc, etc. How many feet does the bird fly before the bicyclists collide?