QR Decomposition

- Householder transformations are a good way to zero out subdiagonal elements of a matrix.
- *A* is decomposed:

$$Q^T A = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad \text{or} \quad QQ^T A = A = Q\begin{bmatrix} R \\ 0 \end{bmatrix}$$

- where $Q^{\mathrm{T}}=H_n\ldots H_2H_1$ is the orthogonal product of Householders and *R* is upper triangular.
- Overdetermined system $Ax=b$ is transformed into the easy-to-solve

$$\begin{bmatrix} R \\ 0 \end{bmatrix} x = Q^T b$$

- Cost: $4mn+4(m-1)(n-1)+\ldots 4(m-n) = O(mn^2)$

Rank Revealing QR

- for i = 1 : n,
  - Among columns i : n of the current **A** matrix, compute column norms
  - If, max column norm is below threshold stop
  - Else, move column with largest norm to the ith column, and store permutation
    - Perform Householder reflection to put zeros below row i in column i of A.
    - Use Householder to transform rest of the matrix A
- end for
- In Matlab, [Q,R,P] = qr(A,0) produces a RRQR
- Cost of RR-QR slightly higher than just QR because of the norm computation of columns at each step

Other Norms

- Here we fit using the "least-squares" or L_2 norm
- Could minimize the residual in other norms
- For example we may have more confidence in some data, and want to be sure that their residual is lower
  - Attach a weight to each residual $$\|r\|_w^2 = \sum_1^m w_i r_i^2$$
- Leads to the problem
- $A^tWAc=A^tW^{1/2}y$ or $(A^tW^{1/2})(W^{1/2}A)\,c = (A^tW^{1/2})\,y$
- Can be solved via normal equations or preferably via QR decomposition of the matrix $W^{1/2}A$
- Or we may like the 1-norm or infinity norm better

$$\|r\|_1 = \sum_1^m |r_i| \qquad \|r\|_\infty = \max_i |r_i|$$

Eigen Values of a Matrix

- Definition:
- A $N\times N$ matrix **A** has an eigenvector **x** (non-zero) with corresponding eigenvalue λ if

$$\mathbf{Ax}=\lambda\,\mathbf{x}$$

- This means

$$\mathbf{Ax}-\lambda\,\mathbf{x}=0 \qquad (\mathbf{A}-\lambda I)\,\mathbf{x}=0$$

- If two numbers multiply to zero one of them is zero
- If a matrix vector product gives a zero vector, then either the vector is zero, or the matrix has zero determinant (is singular).

Solving for eigenvalues

- The zero vector is not an eigenvector (nothing special about $\mathbf{A}0=0$)
- So we need $(\mathbf{A}-\lambda \mathrm{I})\,\mathbf{x}=0 \qquad \|\mathbf{x}\|_2 \neq 0$

$$\det(\mathbf{A}-\lambda\,\mathbf{I})=0$$

- Evaluating the determinant we get an Nth degree polynomial equation, which can be solved for N roots
  - Could be solved numerically using zero finding algorithms
- So a $N \times N$ matrix has at most N eigenvalues

Characteristic Equation

- $A\mathbf{x} = \lambda\,\mathbf{x}$ can be written as

$$(A-\lambda\, I)\mathbf{x} = \mathbf{0}$$

  which holds for $x \neq 0$, so $(A-\lambda\, I)$ is singular and

$$\det(A-\lambda\, I) = 0$$

- This is called the characteristic polynomial. If A is $n \times n$ the polynomial is of degree n and so A has n eigenvalues, counting multiplicities.
- Eigenvalues need not be distinct.
  - E.g. eigenvalues of identity matrix are given by solution of

$$(1-\lambda)^n = 0$$

- So the matrix has N repeated eigenvalues equal to 1

Example

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad (4-\lambda)(2-\lambda) - (1)(3) = 0$$

$$\lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda - 1) = 0$$

- Hence the two eigenvalues are 1 and 5.

Example (continued)

- Once we have the eigenvalues, the eigenvectors can be obtained by substituting back into (A- λ I)**x** = **0**.

$$A - \lambda_1 I = \begin{pmatrix} 4-5 & 3 \\ 1 & 2-5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} 4-1 & 3 \\ 1 & 2-1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$$

- Eigen vectors should give a zero vector when multiplied.
- This gives eigenvectors $(1\ -1)^T$ and $(1\ 1/3)^T$
- Note that we can scale the eigenvectors any way we want.

Assorted properties of eigenvalues & eigenvectors

- Shift eigenvalues of a matrix by τ.
  - Let

$$\mathbf{Ax}=\lambda\ \mathbf{x}$$

  - Add $-\tau$ **x** to both sides

$$(\mathbf{A}-\tau\ \mathbf{I})\mathbf{x}=(\lambda-\tau)\ \mathbf{x}$$

  - We get a new matrix

$$\mathbf{B}=(\mathbf{A}-\tau\ \mathbf{I})$$

  - Shifted eigenvalue $(\lambda-\tau)$
  - Same eigenvector **x**
- Eigenvectors are not in general normalized:
  - If **x** is an eigenvector so is α **x.**
  - Often in software we may normalize eigenvectors to have $\|\mathbf{x}\|_2=1$
- The term *eigenvalue* is a partial translation of the German "eigenvert." A complete translation would be something like "own value" or "characteristic value" .

Eigenvalues and eigenvectors

- Recall a $N \times N$ matrix maps N dimensional vectors to other N dimensional vectors
  - In general it maps elements in R^N to other elements in R^N
- Eigenvectors and eigenvalues provide basic information about this mapping
  - Identify special vectors which remain untransformed (or are just scaled)
- Important in many areas
  - Quantum mechanics – energy levels
  - Acoustics – fundamental frequencies of drums or columns
  - Stability theory – resonant frequencies or critical values of parameters

Eigen-value decomposition

- Represent the matrix in terms of its eigenvalues and eigenvectors
- A $N \times N$ matrix has N eigenvalues and eigen vectors
- Write the N equations

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

- by stacking the vectors $\mathbf{x}_i$ as columns of a matrix $\mathbf{X}$ and the constants λ_i along the diagonal of a matrix
- We get

$$\mathbf{A}\mathbf{X} = \mathbf{X}\Lambda$$

- If all eigenvectors are independent, then $\mathbf{X}^{-1}$ exists, and so

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{X}^{-1}\mathbf{X}\Lambda = \Lambda$$

$$\mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{-1}$$

- This is the **eigenvalue decomposition** of a matrix **A**

Left and Right Eigenvectors

- So far we just talked about matrix products

$$Ax = \lambda x$$

- For a N $\times$ N matrix we can also define a left matrix product

$$y^t A = v$$

- So if we have

$$y^t A = \lambda y$$

then y is a left eigenvector of A

- If A is symmetric $A = A^t$
- $(Ax)^t = x^t A^t = x^t A$
- So left and right eigenvectors of a symmetric matrix are the same

Symmetric Matrices

- A matrix is symmetric if its transpose is equal to itself
- Eigenvalues and Eigenvectors of a real symmetric matrix are real and eigenvectors are orthogonal.
- A matrix A is *positive definite* if for every nonzero vector $\mathbf{x}$ the quadratic form $\mathbf{x}^t A \mathbf{x} > 0$.
- If A is positive definite and λ and $\mathbf{x}$ are an eigenvalue/eigenvector pair, then:

$$A\mathbf{x} = \lambda\, \mathbf{x} \qquad \mathbf{x}^t A \mathbf{x} = \lambda\, \mathbf{x}^t \mathbf{x}$$

- Since $\mathbf{x}^t A \mathbf{x}$ and $\mathbf{x}^t \mathbf{x}$ are both positive it follows that λ is positive.
- If A is a positive definite matrix then:
  - A is nonsingular.
  - The inverse of A is positive definite.
  - Gaussian elimination can be performed on A without pivoting.
  - The eigenvalues of A are positive.

Use of the eigenvalue decomposition

- Can use it to study the properties of A
- Recall condition number definition

$$\operatorname{cond}(A) = \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$$= \frac{\text{maximum stretch}}{\text{maximum shrink}} = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

- Natural frequencies of the matrix
- Powers of a matrix

$$\mathbf{Ax} = \lambda\, \mathbf{x}$$

$$\mathbf{A}(\mathbf{Ax}) = \mathbf{A}\,(\lambda\, \mathbf{x}) = \lambda\, \mathbf{Ax} = \lambda^2\, \mathbf{x}$$

$$\mathbf{A}^n\, \mathbf{x} = \lambda^n\, \mathbf{x}$$

- Apply same idea to EVD

$$\mathbf{A}^n = \mathbf{X}\Lambda^n\mathbf{X}^{-1}$$

Example

- Let A = [-149 -50 -154; 537 180 546; -27 -9 -25]

  This matrix was constructed in such a way that the characteristic polynomial factors nicely.
- $\det(A-\lambda I) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$

- Consequently the three eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and Λ = [1 0 0; 0 2 0; 0 0 3]
- The matrix of eigenvectors is

  X =[1 -4 7 ; - 3 9 – 49; 0 1 91]
- It turns out that the inverse of X also has integer entries.

  X^{-1} = [130 43 133 ; 27 9 28 ; -3 -1 -3]
- These matrices provide the eigenvalue decomposition of our example $A = X\Lambda X^{-1}$

Eigshow

- Eigen values of 2× 2 matrix represent transformations in the plane
- Ideas of symmetry