Optimization:

This homework will study the optimization of the following two functions $f_1(x)$ and $f_2(x)$:

$$f_1(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^t A\boldsymbol{x} - \boldsymbol{b}^t\boldsymbol{x} + c$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \quad c = 0$$

$$f_2(\boldsymbol{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \qquad \text{(Rosenbrock's function)}$$

1. Find the extrema of these functions using calculus. Find the minima
2. Compute the Jacobian and Hessian of each function. Write a Matlab function that computes these.
3. Write a function for implementing the method of steepest descent with a specified tolerance `tol` and a maximum number of iterations `itmax`. You can use the built-in Matlab function `fzero` for line minimization.
4. Compute using the method of steepest descent, setting `tol` to $10^{-6}$ and `itmax` to 100, the minimum for $f_1$ and $f_2$, and an initial guess of $[1, 0.3863]^T$
5. For the Rosenbrock function, implement Newton's method starting from the same guess. Take full steps, without line search.
6. For the Rosenbrock function, implement the limited memory Davidson Fletcher Powell quasi-Newton method starting from the same guess. Use the update formula
$$C^{(k+1)} = C^{(k)} - \frac{C^{(k)}y^{(k)}y^{(k)T}C^{(k)}}{y^{(k)T}y^{(k)}} + \frac{s^{(k)}s^{(k)T}}{y^{(k)T}s^{(k)}}, \qquad C^{(0)} = I$$
   a. Consider $k=2$. What vectors would you store to be able to form $C^{(3)}\mathbf{v}$ for an arbitrary vector $\mathbf{v}$?
   b. How many operations and how much memory would it take to form $C^{(3)}\mathbf{v}$?
   c. Do by hand two iterations for the Rosenbrock function starting from a guess $[1,1]^T$
7. For the Rosenbrock function, using the method of Lagrange multipliers, compute by hand the constrained minimum, subject to the constraint
$$x_1{}^2 + x_2{}^2 = 1.$$