Suppose V has complex shape

- Include V in a domain W that is simply sampled randomly
- Need a function which can verify if a sample is inside V or not
- evaluate f for points inside
- Set it equal to zero for points outside
- Try to make W enclose V as closely as possible
  - E.g. find axes and rotate a rectangular region W to enclose V
- Error estimate is based on the points actually inside in this case

Importance sampling

- Take integrand *f*
- Write it as a product of a function *g* and *f/g*
- Goal: *h=f/g* should be almost constant
- Requirement: *g* should be positive

$$\int_V f dV = \int_V g \frac{f}{g} dV = \int_V hg dV$$

- Choose *g* to be a probability density function $\int_V pdV = 1$

$$I = \int_V \frac{f}{p} p dV \approx V \left\langle \frac{f}{p} \right\rangle \pm V \sqrt{\frac{\langle f^2/p^2 \rangle - \langle f/p \rangle^2}{N}}$$

- So we generate samples from the probability distribution functions of *p*

Stratified sampling

- Run the sampling in a hierarchical fashion
  - Note: this is a standard idea in numerical methods
  - If you need to do something N times – why not use information you get along the way
- Basic idea: divide the volume V into m subregions
- Perform Monte Carlo integration over each subregion
- Add points to regions that have larger variance
- Can show overall variance is reduced
- Note: in high dimensions doing hierarchies is expensive

Mixed approaches

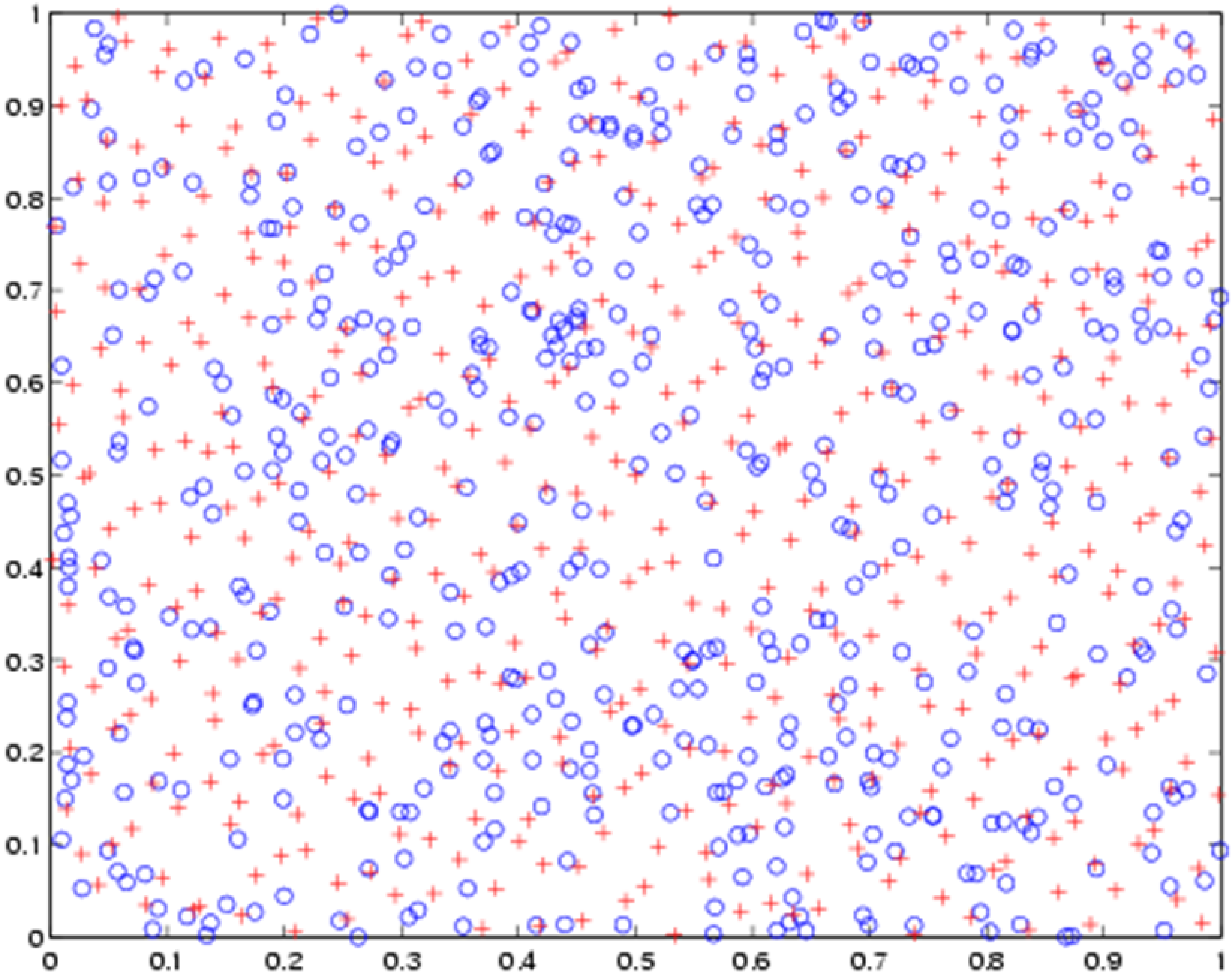
- Importance sampling requires knowledge of a p that is close to f
- We may not know f
- Use hierarchical stratified sampling to build an estimate f with a smaller number of points in each subdomain
- Build a p that looks like f

Quasi Random numbers

- In many applications we need truly random numbers that are independent and identically distributed (iid)
- However this means that we have a high probability of sampling quantities repeatedly
- Go to example
- And moreover we may correspondingly undersample other areas!
- In fact mathematicians have a term for this – "discrepancy"
  - Consider a subset of the domain of volume α
  - Count number of points in this volume, m
  - Discrepancy = $\alpha N - m$

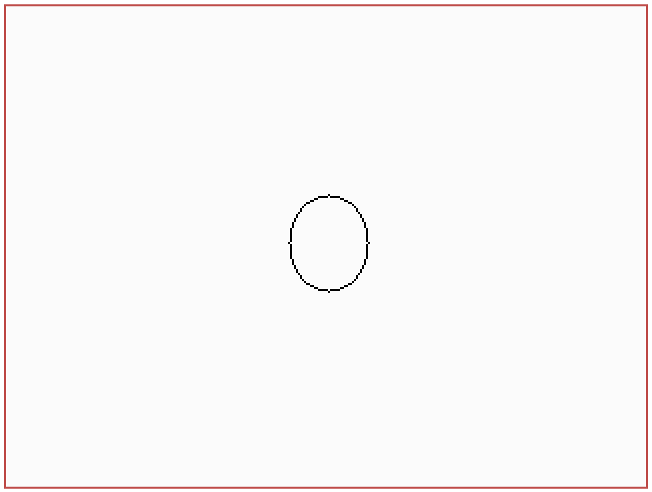
Quasi-Random sequences

- Want to pick sample points "at random", yet spread out in some self-avoiding way
- Sequences of k-tuples that fill k-space more uniformly than pseudo-random points
- Improve asymptotic complexity of search and well spread in multiple dimensions
- Sampling error decreases as $O(N^{-1})$ as opposed to $O(N^{-1/2})$ for pseudo-random



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Synthetic experiments



- Track ellipse with fixed aspect ratio
- 3-D state space (x_c, y_c, σ)
- Each dimension driven by an independent 2nd order harmonic oscillator $x_t = a_1 x_{t-1} + a_2 x_{t-2} + b\omega_t$
- 50 Monte Carlo trials, 500 frame sequence
- Run experiments with pseudo- and quasi-random points and compare errors

Synthetic experiments