MATH 3160 - Probability - Fall 2013

Conditional Probability Lecture Problems

This handout lists the problems in conditional probability & independence (Chapter 3 in Ross) that are covered in lectures from F 9/13 through M 9/23. Some of the problems are taken from Ross, while others aren't. It is your responsibility to learn how to solve these problems, by regularly attending the lectures and asking questions (of me, of your classmates, or on Piazza). **There will not be a separate solution sheet to accompany these problems.**

A problem with an asterisk (*) indicates a classic exercise which has found its immortal place in the study of probability theory and applications.

[UPDATE 09/23/13: Problems L, O and P have been slightly revised. Due to the lack of time, we will skip problems I and S, so they will not appear on the exam. Problems J and R, while not discussed in lecture, are actually variants of the problems we have discussed. If you are interested in their solutions, read the corresponding sections in Ross.]

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A. Suppose 100 men and 200 women are in a room. 40 of the men have an iPhone, while 60 of the women have an iPhone. If a person from this room is chosen at random:

- P(the person has an iPhone) = ?
- P(the person has an iPhone, given that the person is a man) = ?

B. (Recall HW2, Problem 2.23.) Suppose you roll a fair die twice. Let E be the event that the second roll yields a high number than the first roll, and let F_i ($i = 1, 2, 3, 4, 5, 6$) be the event that the first roll yields the number i.

- For each i, find $P(E|F_i)$.
- Show that $P(E) = \sum_{i=1}^{6} P(E|F_i)P(F_i)$. (Use the axioms of probability to justify this expression.)

C. ([Like Example 3.3d] Disease testing.) Suppose a test for HIV is 98% accurate in either direction (*i.e.*, there's a 2% chance that an HIV carrier gets misdiagnosed as not having HIV, and likewise, there's a 2% chance that a non-HIV carrier receives a "false positive" from the test). It is known that 0.5% of a given population carries HIV. You pick a person at random from this population, and administer the HIV test to him. If the test comes back positive, what is the probability that this person *actually* carries HIV?

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D. (Given $P(E|F)$, infer $P(F|E)$.) According to a recent survey conducted within a Manhattan investment bank, 40% of the bank's employees carry an iPhone, and 32% carry an Android phone. Furthermore, it is found that 20% of those who have an iPhone also have an Android phone. Suppose you find, at random, an employee of the bank who carries an Android phone. What is the probability that he also carries an iPhone?

E. ([Like Example 3.2f] Urn problem.) Suppose an urn contains 8 white balls and 6 red balls. We draw 2 balls from the urn without replacement. What is the probability that both balls drawn are red, under the assumption that:

(a) At every draw, each ball in the urn is equally likely to be chosen?

(b) At every draw, each red ball in the urn is twice as likely to be chosen as each white ball in the urn?

F. ([Example 3.2g] The matching problem revisited.) Recall the setup of the matching problem: N people are in a room and place their hats in a black box. Upon mixing the hats, each person picks up a hat from the box at random. Let E_i ($i = 1, 2, \cdots, N$) be the event that person i picks up his own hat. As was discussed in Example 2.5l, the probability that **none of the N people** picks own his own hat is

$$R_N := P\left(\bigcap_{i=1}^{N}(E_i)^c\right) = \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^N \frac{1}{N!} = \sum_{i=2}^{N}(-1)^i \frac{1}{i!}.$$

This result was obtained via the inclusion-exclusion identity.

Now let us ask a deeper question: What is the probability that **exactly k people** ($0 \le k \le N$) pick up their own hats? [In class we set $N = 6$, $k = 3$.] Do this in two steps:

(a) Consider the event that persons 1 through k pick up their own hats, while persons $k+1$ through N fail to pick up their own hats. In other words, compute

$$P\left(\left(\bigcap_{i=1}^{k} E_i\right) \cap \left(\bigcap_{j=k+1}^{N} (E_j)^c\right)\right) = P\left(\bigcap_{i=1}^{k} E_i\right) \cdot P\left(\bigcap_{j=k+1}^{N} (E_j)^c \,\middle|\, \bigcap_{i=1}^{k} E_i\right),$$

where the definition of conditional probability is used. Write out the answer for each of the two terms on the right-hand side, then multiply. [*Note:* To find the first term, you may use the multiplication rule for conditional probabilities.]

(b) In Part (a) we've specified a specific k-tuple of people who get their own hats. Now calculate how many ways you could pick this k-tuple of people from a total of N people. (Notice that any two distinct choices of the k-tuple are mutually disjoint events.) Combine this information with your answer to Part (a), find an expression for the probability that exactly k people pick up their own hats.

G. ([Example 3.3a] Insurance.) An insurance company believes that people can be categorized as being either accident prone or not accident prone. Its statistics reveal that an accident-prone person (resp. a non-accident-prone person) will have an accident at some time within a fixed 1-year period with probability 0.4 (resp. 0.2). Suppose that 30% of the population is accident-prone.

(a) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

(b) Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he/she is accident prone?

H. ([Example 3.3c] Guessing on the SAT.) In answering a multiple-choice section of the SAT, Cartman either knows the answer or guesses. Let p be the probability that he knows the answer, and $1 - p$ be the probability that he guesses. Assume that his guesses are correct with probability $\frac{1}{5}$. What is the conditional probability that Cartman knew the answer to a question, given that he answered it correctly?

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I. (SKIP [Example 3.3h(a)] Chip game.) The player initially has n red chips, while the "bank" has n blue chips. The game proceeds as follows: the player randomly discards a chip in his possession, then claims a chip from the bank (provided that the bank still has chips available). This process continues until all $2n$ chips are discarded. What is the probability that the final chip discarded is red?

J. ([Example 3.3l] How to cheat money from your friend who doesn't know probability.) Suppose we have 3 coins which are identical in form, except that both sides of the first coin are colored red, both sides of the second coin are colored black, and one side of the third coin is colored red and the other side colored black. The 3 coins are mixed in a hat, and 1 coin is randomly selected and put down on the table. If the upper side of the chosen coin is colored black, what is the probability that the other side is colored red?

K. ([Like Example 3.3n] Light bulbs.) A certain hardware store in town sells 3 types of light bulbs. The probability that a type 1 (resp., type 2, type 3) light bulb will last more than 1000 hours of use is 0.6 (resp., 0.4, 0.3). Suppose that 25% of the light bulbs sold are type 1, 35% are type 2, and 40% are type 3.

(a) What is the probability that a randomly selected light bulb will last more than 1000 hours of use?

(b) Given that a light bulb lasted over 1000 hours of use, what is the conditional probability that it was a type j light bulb, where $j = 1, 2, 3$?

L. ([Like Example 3.4e] (Non-)independence of dice throwing events.) Two fair dice are thrown. Let

$$\begin{aligned} E_8 &= \text{event that the sum of the dice is 8,} \\ E_7 &= \text{event that the sum of the dice is 7,} \\ F &= \text{event that the first die equals 3,} \\ G &= \text{event that the second die equals 5.} \end{aligned}$$

For each $i \in \{7, 8\}$, are events E_i and F mutually independent? What about E_i and G? F and G? Finally, are E_i, F, and G independent from one another?

M. ([Binomial probability distribution, Example 3.4f] Flipping coins *ad infinitum*.) An infinite sequence of independent trials of coin flipping is performed. Each flip results in a head with probability p and a tail with probability $1 - p$ ($0 \leq p \leq 1$). What is the probability that:

(a) at least 1 head appears in the first n flips?

(b) exactly k heads appear in the first n flips?

(c) all heads appear?

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N. ([Example 3.4i] Coupon collecting.) There are n types of coupons, and each new one collected is independently of type i with probability $p_i \in [0, 1]$, $\sum_{i=1}^{n} p_i = 1$. Suppose k coupons are to be collected. Let A_i be the event that there is at least one type i coupon among those collected. Compute the following probabilities:

(a) $P(A_i)$.

(b) $P(A_i \cup A_j)$, $i \neq j$.

(c) $P(A_i \cap A_j)$, $i \neq j$.

*O. ([Example 3.4j] Problem of the points.) Independent trials resulting in a success with probability p and a failure with probability $1 - p$ are performed. What is the probability that n successes occur before m failures?

Pascal and Fermat each provided his solution to this problem. Their key insights are presented below. See if you could reconstruct the full solution based on their insights. For concreteness, you may wish to take $n = 4$ and $m = 4$ (think of a best-of-7 series in the MLB, NBA, or NHL playoffs).

(Pascal) Let $P_{n,m}$ denote the probability that n successes occur before m failures. Using conditioning, find a three-term recurrence relation involving $P_{n,m}$, $P_{n,m-1}$, and $P_{n-1,m}$. Then solve this recurrence relation using the obvious (why?) boundary conditions $P_{n,0} = 0$, $P_{0,m} = 1$.

(Fermat) The following statements are equivalent:

(i) n successes occur before m failures.

(ii) At least n successes occur in the first $m + n - 1$ trials.

*P. ([Example 3.4l] Gambler's ruin involving a fair coin.) Two gamblers, A and B, bet on the outcomes of successive flips of a fair coin. On each flip, if the coin comes up heads (resp. tails), then A collects 1 token from B (resp. B collects 1 token from A). The game continues until one player has run out of all tokens. Assume that each successive flip of the coin is independent.

If A starts the game with 15 tokens, and B starts with 10 tokens, what is the probability that A ends up with all the tokens?

[*Note:* This is slightly simplifed from the example in Ross. Also see pp. 8-9 of Prof. Bass' lecture notes for a simple variation of the same problem. For various other scenarios, you may replace the fair coin by a biased coin (with probability p of landing a head), and/or change the the number of tokens given to each player initially.]

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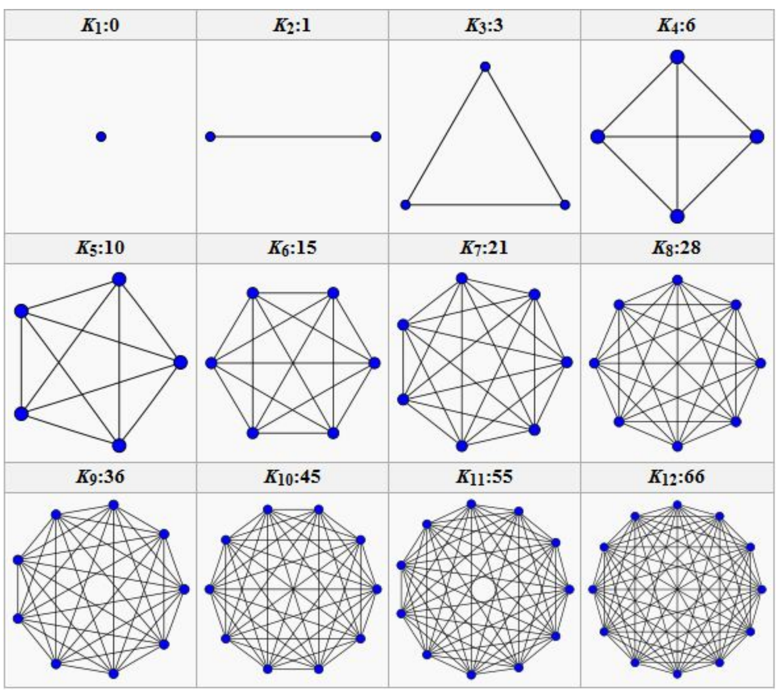
*Q. ([Example 3.4m, of interest to CS majors] 2-coloring of a complete graph; the *probabilistic method* of Erdős.) A **complete graph** is a graph in which every distinct pair of vertices is connected by an edge. Let us denote by K_n the complete graph of n vertices (see figure); there are $\binom{n}{2}$ edges in K_n. (Why?)



Now let us color each of the $\binom{n}{2}$ edges in either red or black. There are a finite number of ways to "2-color" all edges of K_n. For a fixed integer $k \leq n$, answer the following question:

Does there exist a 2-coloring of the edges so that no set of k vertices has all of its $\binom{k}{2}$ connecting edges the same color?

Show that the answer is affirmative if n is *not too large*. Even better, give an upper bound on what n can be to affirm the condition.

R. ([Example 3.5b] Paternity testing.) [*Disclaimer:* The two versions below are 100% identical in mathematical content. If you are unfamiliar with Maury Povich's show, or feel generally offended by it, then read the sanitized version only.]

Sanitized version. A female chimp has given birth. However, it is not certain which of the two male chimps is the father. Before any genetic test has been done, it is believed that the probability that male chimp #1 (resp. #2) is the father is p (resp. $1-p$). DNA obtained from the mother and the two male candidates indicate that, on one particular location of the genome, the mother has the gene pair (A, A), male chimp #1 has the gene pair (a, a), and male chimp #2 has the gene pair (A, a). If a DNA test shows that the baby chimp has the gene pair (A, a), what is the probability that male chimp #1 is the father?

Dramatized version. A woman goes on the Maury show to find out which of her two former boyfriends, Adonis or Brutus, is the father of her newborn child. Prior to the revelation of the paternity test, the vocal show audience agree that with probability p (resp. $1-p$), Adonis (resp. Brutus) is the father. DNA obtained from the woman and the two men indicate that, on one particular location of the genome, the woman has the gene pair (A, A), Adonis has the gene pair (a, a), and Brutus has the gene pair (A, a). If a DNA test shows that the child has the gene pair (A, a), what is the probability that Maury declares to Adonis, "YOU ARE THE FATHER!"?

S. (SKIP [Example 3.5d] The matching problem, encore.) At a party, N women take off their hats. The hats are then mixed up, and each woman picks up a hat at random. We say that a match occurs if a woman selects her own hat. What is the probability that

(a) no matches occur?

(b) exactly k matches occur?

Recall Problem F of this handout for one approach to the problem. Here we present a different approach. Let E stand for the event that no matches occur. To be explicit, we write P_N for the probability of the matching problem involving N women. So for instance, $P_N(E)$ is the probability that no matches occur amongst the N women. The event we wish to condition upon is M (resp. M^c), the event that the first woman selects her hat (resp. the first woman does not select her hat). Clearly

$$P_N(E) = P_N(E|M)P_N(M) + P_N(E|M^c)P_N(M^c).$$

Using this identity, find a three-term recurrence relation involving $P_N(E)$, $P_{N-1}(E)$, and $P_{N-2}(E)$. Then proceed to solve the problem.